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CALCULUS

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CALCULUS

BY

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PREFACE

The first part of this book is a text-book of elementary calculus and contains the material usually given in a one year course in the calculus. It aims to present the underlying notions of the subject in a manner intelligible to beginners but with precision, to aid the student to master the technique of the calculus, and by a variety of applications to give him a realization of its usefulness and power.

Except for brief chapters on determinants and solid geometry, the remainder of the book is devoted to topics of advanced calculus, including the theory of partial differentiation and some of its more important and interesting applications, differential equations, line and surface integrals, the development of implicit functions in series by the method of successive approximations with applications to curve tracing, and Fourier series. It closes with a chapter on functions of a complex variable. In the treatment of this material I have endeavored not to assume a higher degree of mathematical maturity than a student in his second year of calculus is likely to possess. I have supplemented the theoretical discussion by numerous exercises, some merely illustrative, others requiring independent thought.

Throughout the book I have sought to present the theory with as little formality as possible. The proofs are based on the fundamental theorems respecting the existence of limits and the properties of continuous functions. These theorems are formulated and illustrated geometrically early in the book, but their arithmetical proofs are postponed to a chapter on numbers and continuous functions near the end.

The book has taken shape gradually in pamphlets which for several years past it has been my practice to prepare for the use of classes at Princeton. In writing it I have profited by suggestions from other books on the subject, especially the *Cours d'Analyse* of de la Vallée Poussin. I am also indebted to my colleague Professor Wedderburn for many valuable suggestions and criticisms and for aid in seeing the book through the press.

HENRY B. FINE

PRINCETON, September, 1927

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I. VARIABLES. LIMITS. FUNCTIONS

1. The real numbers and the points of a line. Take a horizontal line and on it a fixed point O as origin; also a unit segment OI for measuring lengths or distances. We may represent any given positive number, a, rational or irrational, by the point A which is at the distance a to the right of O; and $-a$ by the corresponding point A' to the left of O.

FIG. 1.

We thus obtain a picture of the relation of order which exists among the real numbers. To two numbers a and b such that $a < b$ correspond two points A and B such that A is to the left of B.

2. Number intervals. The set of all real numbers between a and b, a and b themselves included, is called the *number interval* a, b. It is represented by the symbol (a, b). We call $b - a$ the *length of* (a, b).

3. Variables. A constant, as c, is a symbol for a single number. A *variable*, as x, is a symbol for any one of a set of numbers. If the set consists of all the real numbers, or of all in an interval (a, b), then x is called a *continuous variable.*

We often think of a variable as running through the numbers of its set in some definite order. We may call the set as thus arranged, the *range* of the variable. In particular,

(1) The range may be a never-ending sequence of isolated numbers, as when x is supposed equal to $1/2, 2/3, 3/4, 4/5, \cdots$ successively.

(2) The range may be a number interval. Thus in Fig. 1 let X denote any point between A and B, and x the corresponding number. If X move along the line from A to B, x will equal successively all numbers between a and b. A variable which is supposed thus to change from one number to another through all intermediate numbers is said to *vary continuously.*

4. Limit of a variable. As x runs through the sequence $1/2, 2/3, 3/4, \cdots$, it approaches 1 in such a manner that the difference $1 - x = 1/2, 1/3, 1/4, \cdots$ will ultimately become and remain less than .01, or .001, or any other positive number that may be assigned, however small. To indicate this, we say: "x approaches 1 as limit." In general

The variable x is said to approach the constant c as limit, if it be true for every positive number δ that may be assigned, however small, that, as x runs through its range, the difference $x - c$ will ultimately become and remain numerically less than that number δ.

To indicate that x approaches the limit c, we write

$$x \to c \qquad \text{or} \qquad c \doteq \lim x \tag{1}$$

The symbol $|a|$ means the numerical value of a. Thus $|-3| = |3| = 3$. Hence, using δ to denote any given positive number whatsoever, however small, we may write:

$$x \to c \quad \textit{if} \quad |x - c| \textit{ ultimately remains } < \delta \tag{2}$$

EXAMPLE. What limit, if any, does x approach as it runs through each of the following ranges?

1. $3, 2\frac{1}{2}, 2\frac{1}{4}, 2\frac{1}{8}, \cdots$ → 2
2. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots$
3. $.3, .33, .333, \cdots$ →
4. $1, 1\frac{1}{2}, 1, 1\frac{1}{4}, 1, 1\frac{1}{8}, \cdots$
5. $1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{8}, \cdots$ → 0
6. $1, 2, 3, 4, \cdots$

Respecting the *numerical values of sums and products*, observe that

$$|a + b| \leqq |a| + |b| \qquad |ab| = |a|\,|b| \tag{3}$$

Thus, $|-2 + (-3)| = |-2| + |-3|$, $|-2 + 3| < |-2| + |3|$,

$$|-3(-2)| = |-3| \cdot |-2|$$

5. On the existence of a limit. Not every variable approaches a limit. Thus $x = 1, 2, 3, 4, \cdots$ does not. Also $x = 1, 2, 1, 2, \cdots$ does not. But, as will be shown later, in the following two cases we may always conclude that a limit *exists*, though the actual determination of its value may be difficult:

1. *If x continually increases but remains less than some finite number c, it approaches a limit; and this limit is either c or some lesser number.*

2. *If x continually decreases but remains greater than some finite number c, it approaches a limit; and this limit is either c or some greater number.*

EXAMPLE. A circle of radius r being given, let x denote the area of an inscribed regular polygon of n sides. As n increases, x increases but remains always less than the area of any circumscribed polygon; hence, by 1, it approaches a limit l. Similarly, by 2, the area y of the corresponding circumscribed polygon approaches a limit l'. It can be proved geometrically that $l = l' = r^2\pi$ ($\pi = 3.14159 \cdots$). This number is called the area of the circle.

6. Infinitesimals. A variable which approaches 0 as limit is called an *infinitesimal*. Thus $x = 1, 1/2, 1/3, 1/4, \cdots$ is an infinitesimal.

Evidently *the difference between any variable which approaches a limit and that limit is an infinitesimal.* If $x \to c$, we may set $x - c = h$ and then write $x = c + h$, $h \to 0$.

Consider the following examples:

EXAMPLE 1. Prove that $3h + 2k \to 0$ when $h \to 0$ and $k \to 0$.

Let δ denote any given positive number, however small. It is to be proved that as h and k approach 0, $|3h + 2k|$ ultimately becomes and remains $< \delta$. But by § 4 (3), $|3h + 2k| \overline{\gtrless} |3h| + |2k|$. Hence we shall have $|3h + 2k| < \delta$ when $|3h| < \delta/2$ and $|2k| \lhd \delta/2$ $\therefore$ when $|h| < \delta/6$ and $|k| < \delta/4$.

Thus $|3h + 2k| < .01$ when $|h| < .0016$ and $|k| < .0025$; and so on. Hence $3h + 2k \to 0$.

EXAMPLE 2. Prove that $(3-k)h+(4+h^3)k \to 0$ when $h \to 0$ and $k \to 0$.

Since $h, k \to 0$, we may begin by restricting h, k to values such that $|h|, |k| < 1$, and therefore $|3-k|, |4+h^3| < 5$. We then have $|(3-k)h+(4+h^3)k| < \delta$ when $5|h|+5|k| < \delta$ $\therefore$ when $|h|, |k| < \delta/10$. But $h, k \to 0$. Hence however small δ be taken, $|h|, |k|$ ultimately remain $< \delta/10$.

Therefore $(3-k)h+(4+h^3)k \to 0$ when $h \to 0$ and $k \to 0$.

These examples illustrate the following theorem:

Let h and k be infinitesimals, and let P and Q denote constants, or variables for which positive numbers c and C can be found such that $|P|, |Q| < C$ when $|h|, |k| < c$. Then

$$Ph + Qk \to 0 \text{ when } h, k \to 0 \tag{1}$$

For restricting h, k to values such that $|h|, |k| < c$, we have $|Ph+Qk| < \delta$ when $C|h|+C|k| < \delta$ and therefore when $|h|, |k| < \delta/2C$. But $h, k \to 0$. Hence, however small δ is taken, we have ultimately $|h|, |k| < \delta/2C$.

Therefore $Ph + Qk \to 0$ when $h, k \to 0$.

7. Limits of sums, products, quotients. *The limit of the sum of two variables which approach limits is the sum of those limits.*

The limit of the product of two variables which approach limits is the product of those limits.

The limit of the quotient of two variables which approach limits is the quotient of those limits, unless the limit of the divisor is 0.

Let $u \to a$ and $v \to b$. It is to be proved that

1. $(u+v)-(a+b)$ 2. $uv - ab$ 3. $\frac{u}{v} - \frac{a}{b}$ $(b \neq 0)$

all $\to 0$ when $u, v \to a, b$. But, setting $u = a+h, v = b+k$, 1., 2., 3., become

1. $[(a+h)+(b+k)]-(a+b) = h+k$
2. $(a+h)(b+k) - ab = bh + (a+h)k$
3. $\dfrac{a+h}{b+k} - \dfrac{a}{b} = \dfrac{bh-ak}{b(b+k)} = \dfrac{1}{b+k}h - \dfrac{a}{b(b+k)}k$

The last members of 1., 2., 3. are all of the form[1] $Ph+Qk$ of § 6, and therefore $\to 0$ when $h, k \to 0$, that is, when $u, v \to a, b$.

By repeated applications of the first two theorems they may be extended to sums and products of any finite number of variables. In particular,

4. If $u \to a$, then $u^n \to a^n$

EXAMPLE. Let $R(x)$ denote any expression which, like $x^2 + 8x$ or $(2x+1)/x^2$, involves x rationally; and let c be any value of x for which no denominator in $R(x)$ vanishes. It follows from the theorems just proved that

$$\text{when } x \to c \quad \text{then} \quad R(x) \to R(c)$$

Thus, when $x \to c (\neq 0)$, we have

$$\lim \frac{2x+1}{x^2} = \frac{\lim (2x+1)}{\lim x^2} = \frac{2 \lim x + 1}{(\lim x)^2} = \frac{2c+1}{c^2}.$$

8. Limits of roots. *The limit of the nth root of a positive variable which approaches a limit is the nth root of that limit.*

First, let x decrease toward $c (\geqq 0)$ as limit. Then $x^{1/n}$ decreases but remains $> c^{1/n}$ and therefore approaches some limit l (§ 5, 2.).

But when $x^{1/n} \to l$, then $x \to l^n$, (§ 7, 4.). Hence[2] $l^n = c$ and therefore $l = c^{1/n}$.

Similarly (by § 5, 1.) when x increases toward $c (> 0)$ as limit, $x^{1/n} \to c^{1/n}$.

Hence if $x \to c$ through any positive values, then $x^{1/n} \to c^{1/n}$.

9. The symbol $\lim\limits_{x \to c} E(x)$. Let $E(x)$ be an expression in x which has a single real value for each value of x under consideration.

If for all modes of approach of x to c as limit (x remaining $\neq c$), $E(x)$ approaches one and the same number l as limit, this limit l is represented by the symbol $\lim\limits_{x \to c} E(x)$.

[1] In 3., when $|k| < |b|/2$ we have $|P| < 2/|b|$ and $|Q| < 2|a|/b^2$.

[2] It is here assumed that x cannot approach two different limits simultaneously. Prove this.

If x is restricted to values $< c$, or to values $> c$, the symbol $\lim_{x \to c} E(x)$ may be replaced by $\lim_{x \to c} E(x)$, or by $\lim_{x \to c} E(x)$.

Thus, $\lim_{x \to 1} (x^2 + 3x) = 4$, $\lim_{x \to 1} \sqrt{1 - x} = 0$, $\lim_{x \to 1} \sqrt{x - 1} = 0$.

When $E(x)$ is rational with respect to x and has no denominator which vanishes when $x = c$, then, (§ 7, Example)

$$\lim_{x \to c} E(x) = E(c) \tag{1}$$

that is, *the limit approached by $E(x)$ when $x \to c$ is the value which $E(x)$ has when $x = c$.*

10. The limit ∞. If a variable x will ultimately become and remain numerically greater than every given positive number C, however large, we say that it *approaches infinity as limit*,[1] and write $|x| \to \infty$. If x ultimately remains positive, we write $x \to \infty$; if x ultimately remains negative, we write $x \to -\infty$.

Thus if $x = 1, 2, 3, \cdots$, then $x \to \infty$; if $x = -1, -2, -3, \cdots$ then $x \to -\infty$.

11. Theorem 1. *Let a be any constant; then* $\lim_{|x| \to \infty} a/x = 0$

For, however small a positive value is assigned to δ, we have

$$|a/x| < \delta \qquad \text{when} \qquad |x| > |a|/\delta$$

Thus we make $2/|x| < .001$ when we take $|x| > 2/.001 = 2000$.

For $\lim_{x \to \infty} E(x)$ we often write $E(\infty)$; thus $a/\infty = 0$.

EXAMPLE 1. *When $x \to \pm \infty$, then $a_0x^n + a_1x^{n-1} + \cdots + a_n$ approaches infinity with the sign of a_0x^n, its term of highest degree.*

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = x^n(a_0 + a_1/x + \cdots + a_n/x^n),$$

and $$a_1/x, \cdots, a_n/x^n \to 0 \text{ when } x \to \pm \infty.$$

[1] The word "limit" in the phrase "approaches infinity as limit" has a meaning wholly distinct from that in § 4. $x \to \infty$ means merely that no number can be assigned which x will not ultimately exceed.

EXAMPLE 2. If $E(x) = (3x^2 + 4x)/(2x^2 + 1)$, find $\lim\limits_{x \to \infty} E(x)$.

$$\frac{3x^2 + 4x}{2x^2 + 1} = \frac{x^2(3 + 4/x)}{x^2(2 + 1/x^2)} = \frac{3 + 4/x}{2 + 1/x^2} \quad \therefore \lim_{x \to \infty} E(x) = \frac{3}{2}.$$

EXAMPLE 3. Prove that **1.** $\lim\limits_{x \to \infty} \dfrac{2x^4 + x}{x^3 + 5} = \infty$ **2.** $\lim\limits_{x \to \infty} \dfrac{x^2 + 3}{x^3 + 2x} = 0$

$$\textbf{3. } \lim_{x \to \infty} \frac{(1 - x)(3 + x^2)}{(2x^2 + 5)x} = -\frac{1}{2}.$$

12. Theorem 2. *Let a be any constant except 0; then* $\lim\limits_{x \to 0} |a/x| = \infty$

For, however great a number C is assigned, we have

$$|a/x| > C \qquad \text{when} \qquad |x| < |a|/C$$

Thus we make $|2/x| > 1000$ when we take $|x| < 2/1000 = .002$.

If $\lim\limits_{x \to c} |E(x)| = \infty$, we say that $E(x)$ *becomes infinite* when $x = c$, and we write $|E(c)| = \infty$. Thus $|a|/0 = \infty$.

A rational fraction in x, reduced to its lowest terms, becomes infinite for values of x for which its denominator vanishes.

Thus $x^2/(x - 2)$ becomes ∞ when $x = 2$. For when $x \to 2$, x^2 ultimately becomes and remains $> 2^2 - 1 = 3$ $\therefore x^2/|x - 2| > 3/|x-2|$. But $3/|x - 2| \to \infty$. Hence $x^2/|x - 2| \to \infty$.

13. Indeterminate forms. 1. When $x = 1$, the fraction $(x^2 - 1)/(x - 1)$ takes the meaningless form $0/0$. This is because both numerator and denominator contain the factor $x - 1$ which vanishes when $x = 1$. We cannot remove this common factor when $x = 1$ since that would involve division by 0, which is never admissible. But, for $x \neq 1$, we have $(x^2 - 1)/(x - 1) = x + 1$ and therefore

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2$$

We therefore *assign* to $(x^2 - 1)/(x - 1)$ for $x = 1$ the value 2.

2. In general, suppose that when $x = c$ an expression $E(x)$ takes an *indeterminate form*, that is, a form, such as $0/0$,

$\infty \cdot 0$, $\infty - \infty$, which is arithmetically meaningless. *If* $\lim_{x\to c} E(x)$ *exists, we assign to* $E(c)$ *the value* $\lim_{x\to c} E(x)$.

EXAMPLE 1. The expression $E(x) = (2x^2 - 5x + 2)/(x^2 - 3x + 2)$ becomes $0/0$ when $x = 2$. This means that both terms of the fraction contain the factor $x - 2$. Canceling this common factor, we find

$$E(2) = \lim_{x\to 2} \frac{2x^2 - 5x + 2}{x^2 - 3x + 2} = \lim_{x\to 2} \frac{2x - 1}{x - 1} = 3$$

EXAMPLE 2. Find the following limiting values:

1. $\lim_{x\to 3} \frac{x^2 - 4x + 3}{x^2 - 7x + 12}$

2. $\lim_{x\to 1/2} \frac{2x^3 + x^2 + 5x - 3}{8x^3 - 8x^2 + 4x - 1}$

3. $\lim_{x\to 4} \frac{2x^3 - 7x^2 - 3x - 4}{x^3 - 5x^2 - 8x + 48}$

4. $\lim_{x\to 1}\left[\frac{1}{2x^2+x-3} - \frac{1}{3x^2-x-2}\right]$

14. Functions. 1. If y is so related to x that to each value of x in the interval (a, b) corresponds a single real value of y, then y is said to be a *function* of x in (a, b).

Thus, the equation $y = x^2$ defines y as a function of x in the interval $(-\infty, \infty)$. Again $y = \sqrt{1 - x^2}$ defines y as a function of x in the interval $(-1, 1)$.

2. The statement that y is a function of x does not necessarily imply that we can express y in terms of x.

Thus, suppose that t hours after noon on a certain day a particular thermometer records T degrees of temperature. To each value of t corresponds a definite value of T. Hence T is a function of t. But we cannot express T in terms of t.

3. To indicate that y is a function of x, we write $y = f(x)$ and then represent the value which y has when $x = c$ by $f(c)$. When y is given as equal to some expression in x, we use $f(x)$ to represent that expression. Thus, if $y = x^2 + 3x$, then $f(x) = x^2 + 3x$ and $f(2) = 2^2 + 3 \cdot 2 = 10$.

15. Graphs of functions. Let $y = f(x)$ denote some given function of x in the interval (a, b). Taking rectangular axes Ox, Oy, plot the point P whose abscissa is any value of x in (a, b) and whose ordinate is the corresponding value of $f(x)$.

The set of all such points P is called the *graph* of $y = f(x)$ in (a, b). The definition of function, § 14, says nothing as to the arrangement of these points P except that one of them, and but one, lies on each parallel to Oy between aA and bB. But in Fig. 2 it is supposed that they form a smooth unbroken curve arc, APB, such as may be traced by a point moving continuously from A to B. It will be found that the functions $y = f(x)$ which we meet in the Calculus all have graphs of this type — except when $f(x)$ becomes infinite at some point in (a, b), a case illustrated in Fig. 4.

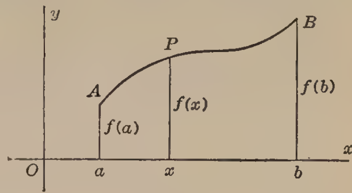

FIG. 2.

16. Continuous functions. Obviously the function $y = f(x)$ pictured in Fig. 2 has the property that small changes in the value of x cause but small changes in the value of $f(x)$. If c denote any point in (a, b), then $f(c)$ has a definite finite value; when $x - c$ is small, so also is $f(x) - f(c)$; and $\lim\limits_{x \to c} [f(x) - f(c)] = 0$. To indicate all this, we say that $f(x)$ is *continuous* at the point $x = c$. In general,

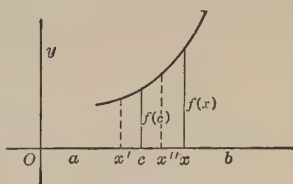

FIG. 3.

A function $f(x)$ is said to be continuous at the point $x = c$ when $f(c)$ has a definite finite value and $\lim\limits_{x \to c} f(x) = f(c)$.

A point where either of these conditions is not satisfied is called a point of discontinuity.

A function $f(x)$ is said to be continuous in the interval (a, b) when it is continuous at every point of the interval, it being sufficient at the end points that $\lim\limits_{x \to a} f(x) = f(a)$, $\lim\limits_{x \to b} f(x) = f(b)$.

By turning to §§ 7, 8, and 12, it will be seen that we have already proved that a polynomial in x is continuous for all finite values of x; a rational fraction in x, reduced to its lowest terms, is discontinuous at points where its denominator vanishes, but continuous for all other finite values of x; the function $\sqrt[n]{x}$ is continuous for all finite positive values of x and for $x = 0$.

17. Points of discontinuity. The only points of discontinuity that immediately concern us are points $x = c$ at which a function $f(x)$ becomes infinite in the sense $\lim_{x\to c} |f(x)| = \infty$. But, as the following examples show, the definition of function, § 14, makes other discontinuities possible.

The condition for continuity, $\lim_{x\to c} f(x) = f(c)$, implies that, any positive number δ being given, however small, it is possible to find an interval (x', x'') containing c as an interior point (Fig. 3), and such that the difference between every two values of $f(x)$ in (x', x'') is $< \delta$. A point c for which this is not the case is a point of discontinuity.

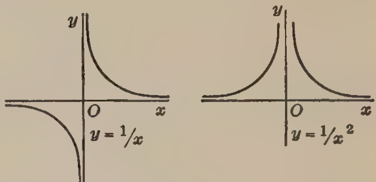

Fig. 4.

Example 1. Both $y = 1/x$ and $y = 1/x^2$ are continuous for all values of x except 0. But $\lim_{x\to 0} 1/|x| = \infty$, $\lim_{x\to 0} 1/x^2 = \infty$. Hence both functions are discontinuous at $x = 0$. In any interval (x', x'') containing 0, both $1/x$ and $1/x^2$ take values whose difference will exceed any given number.

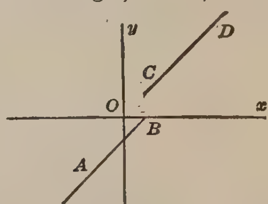

Fig. 5.

Example 2. The function $y = f(x)$, where $f(x) = x - 1$ when $x \leqq 1$, but $f(x) = x$ when $x > 1$, is discontinuous at $x = 1$; for $f(1) = 0$ but $\lim_{x\to 1} f(x) = 1$. In any interval (x', x'') containing 1 as an interior point, $f(x)$ takes values whose difference exceeds 1. The graph consists of the half-lines AB, CD, the point C excluded.

EXAMPLE 3. As $x \to 0$, the function $\sin(1/x)$ continually oscillates between -1 and 1 and therefore does not approach a limit. Hence $\sin(1/x)$ is discontinuous at $x = 0$. Therefore the convention made in § 13, 2, gives no meaning to $\sin(1/0)$.

18. Properties of continuous functions. As will be proved later, it follows from the definition of a continuous function that

1. *If $f(x)$ is continuous in (a, b), then among the different values of $f(x)$ in (a, b) there is a greatest*[1] *value M and a least value m.*

2. *If $f(x)$ is continuous in (a, b), then between every two values x_1 and x_2 of x in (a, b), $f(x)$ takes at least once every value between $f(x_1)$ and $f(x_2)$.*

These theorems are obvious geometrically if we assume that, in (a, b), $y = f(x)$ has a graph of the kind described in § 15.

EXAMPLE. Suppose that the graph of $y = f(x)$ is the curve arc $CDEF$. Here $m = AC$ and $M = GE$. If $x_1 = OX_1$, $x_2 = OX_2$, we have $X_1P_1 = f(x_1)$, $X_2P_2 = f(x_2)$. Let OY' represent any number between X_1P_1 and X_2P_2. The parallel to Ox through Y' will meet the unbroken arc P_1P_2 in some point P', and if the abscissa OX' of P' be x', we have $f(x') = x'P' = OY'$.

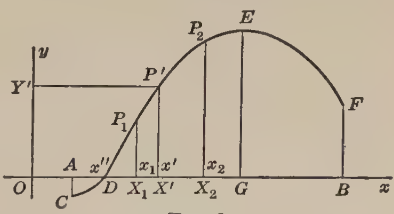

FIG. 6.

Observe that corresponding to the fact that $f(a) = AC$ is $-$, and $f(b) = BF$ is $+$, there is a value of x in (a, b) for which $f(x)$ is 0, namely the value $x'' = OD$.

19. Polynomials in x. This name is given to functions of the form $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, where n is a positive integer and $a_0, a_1, \cdots a_n$ are constants.

[1] It is by no means true of every infinite set of numbers that it contains a greatest and a least number. Thus the set consisting of all numbers x such that $2 < x < 3$ contains no greatest or least number.

A value b of x such that $f(b)$ is 0 is called a *root* of $f(x)$ or of the equation $f(x) = 0$.

1. The following discussion leads us to the most expeditious method of computing the value of $f(x)$ for a given value of x and to the theorem:

$f(x)$ is exactly divisible by $x - b$ when and only when $f(b) = 0$.

Suppose $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ to be divided by $x - b$. Let $\phi(x) = a_0x^2 + C_1x + C_2$ denote the quotient, and R the remainder, C_1, C_2, R being constants whose values are to be determined. Then

$$\begin{aligned} a_0x^3 + a_1x^2 + a_2x + a_3 &\equiv (a_0x^2 + C_1x + C_2)(x - b) + R \\ &\equiv a_0x^3 + (C_1 - a_0b)x^2 + (C_2 - C_1b)x + (R - C_2b) \end{aligned}$$

Equating coefficients of like powers of x,

$$\begin{array}{lll} C_1 - a_0b = a_1 & \therefore C_1 &= a_0b + a_1 \\ C_2 - C_1b = a_2 & \therefore C_2 &= C_1b + a_2 = a_0b^2 + a_1b + a_2 \\ R - C_2b = a_3 & \therefore R &= C_2b + a_3 = a_0b^3 + a_1b^2 + a_2b + a_3 = f(b) \end{array}$$

Hence (1) $f(x) \equiv (x - b)\phi(x)$ *when and only when* $f(b) = 0$; and (2) we may compute C_1, C_2, $f(b)$ successively as in the accompanying scheme: C_1 being got by adding a_0b to a_1, then C_2 by adding C_1b to a_2, and so on.

$$\begin{array}{cccc|l} a_0 & a_1 & a_2 & a_3 & b \\ & a_0b & C_1b & C_2b & \\ \hline a_0 & C_1 & C_2 & R = f(b) & \end{array}$$

This process is called *synthetic division* (or substitution).

EXAMPLE 1. Find the value of $f(x) = 3x^4 - 8x^2 + 10x - 5$ when $x = -2$.

$$\begin{array}{rrrrr|l} 3 & +0 & -8 & +10 & -5 & -2 \\ & -6 & +12 & -8 & -4 & \\ \hline 3 & -6 & +4 & +2, & -9 & \end{array}$$

Hence $f(-2) = -9$

This is the remainder in the division of $f(x)$ by $x - (-2) = x + 2$. The quotient is $3x^3 - 6x^2 + 4x + 2$.

EXAMPLE 2. Prove that 3 and $-1/2$ are roots of

$$2x^4 - 5x^3 + x^2 - 10x - 6 = 0$$

2. For polynomials $f(x)$ with integral coefficients we may deduce the following theorem from the fact that, if b is a root, then $a_0b^n + \cdots + a_{n-1}b + a_n = 0$:

Let the coefficients of $f(x)$ be integers. If an integer b be a root of $f(x)$, then b is a factor of a_n. And if a rational fraction β/α (in its lowest terms) be a root, then β is a factor of a_n and α of a_0.

In consequence of this theorem, the rational roots, if any, may be found by a limited number of trials. If one such root has been found, the search for the rest may be shortened by using the fact that if b be a root of $f(x)$, then, since $f(x) \equiv (x-b)\phi(x)$, the remaining roots of $f(x)$ are the roots of the "depressed polynomial" $\phi(x)$.

The irrational roots, if any, of $f(x)$ or the depressed polynomial $\phi(x)$ may be found approximately by applications of the theorem § 18, 2.:

If $f(a)$ and $f(b)$ have opposite signs, there is a root of $f(x)$ between a and b.

We seek first to locate the roots between consecutive integers, then between consecutive tenths, and so on. The method will be explained later.

EXAMPLE 3. Find the roots of $3x^4 + 4x^3 - 19x^2 - 8x + 12 = 0$.

EXAMPLE 4. Show that $2x^3 - 3x^2 - 9x + 8 = 0$ has roots in the x-intervals $(0, 1)$, $(2, 3)$, $(-2, -1)$.

3. Let $f(x) = a_0x^n + \cdots$ denote a given polynomial of degree n with real coefficients. It is the fundamental theorem of Algebra that every polynomial $f(x)$ has at least one root. By aid of this theorem and the theorem that if $f(b) = 0$, then $f(x) \equiv (x - b)\phi(x)$, where $\phi(x)$ is a polynomial of degree $n - 1$, it can be proved that n numbers β_1, β_2, $\cdots$, β_n exist such that $f(x) \equiv a_0(x - \beta_1)(x - \beta_2) \cdots (x - \beta_n)$.

Evidently $f(\beta_1) = 0$, and so on. It is therefore said that

Every algebraic equation $f(x) = 0$ of degree n has n roots.

Certain of these roots may be equal. Those, if any, which are imaginary occur in pairs of the form $a + bi$, $a - bi$, where a and b are real and $i = \sqrt{-1}$. But

$$[x - (a + bi)][x - (a - bi)] = x^2 - 2ax + (a^2 + b^2).$$

Hence

Every polynomial $f(x)$ with real coefficients is a product of factors of the first or second degree with real coefficients.

20. Rational functions. Any function which can be reduced to a polynomial in x or to a fraction whose numerator and denominator are such polynomials is called a *rational function of* x, *integral or fractional.*

21. Algebraic functions. A function $y = \phi(x)$ is called *algebraic* if it satisfies an *algebraic equation in* x, y, that is, an equation $f(x, y) = 0$ in which $f(x, y)$ is a polynomial in x, y. Thus $y = \sqrt{x}$ is algebraic: it satisfies $y^2 - x = 0$.

A rational function is algebraic. Thus $y = 1/x$ satisfies $xy - 1 = 0$.

Functions, such as $y = \sin x$, $y = \log x$, which are not algebraic, are called *non-algebraic* or *transcendental.*

EXERCISE I

1. If $f(x) = x^5 - x^3 + 3x^2 - x + 4$, find $f(0)$, $f(1)$, $f(-1)$, $f(6)$.

2. If $f(x) = x^m$, then $f(x)f(y) = x^m y^m = (xy)^m = f(xy)$.

3. If $f(x) = a^x$, show that $f(x)f(y) = f(x + y)$.

4. If $f(x) = \log x$, show that $f(x) + f(y) = f(xy)$; also $f(x^n) = nf(x)$.

5. If $f(x) = x + 1/x$, show that $[f(x)]^2 = f(x^2) + 2$; also $[f(x)]^3 = f(x^3) + 3f(x)$.

6. If $y = f(x) = (1 - x)/(1 + x)$, show that $x = f(y)$.

7. Prove by § 7 that, if two functions $f(x)$ and $\phi(x)$ are both continuous at $x = c$, so also are the functions $f(x) + \phi(x)$ and $f(x)\phi(x)$; also $f(x)/\phi(x)$ unless $\phi(c) = 0$.

8. Show that, by the convention in § 13, $f(x) = (x^2 - 1)/(x - 1)$ is continuous at $x = 1$.

9. Show that by the same convention $f(x) = x \sin (1/x)$ is continuous at $x = 0$.

10. At what points, if any, are the following functions discontinuous?

1. $\dfrac{2x + 1}{x^3 - x}$ 2. $\dfrac{(x - 1)(x + 1)^2}{(x - 1)^2(x + 1)}$

3. $\dfrac{x}{x^2 + 4}$ 4. $\dfrac{x^2 - 5x + 4}{x^2 - 7x + 6}$

5. $\dfrac{2 - x}{(4 - x^2)^{1/2}}$ 6. $\dfrac{1}{x^2 - 1} - \dfrac{1}{x(x^2 - 1)}$

11. Find the graph of $y = |x|$. Is this function discontinuous at $x = 0$?

12. Find the graph of $y = 1/(1 - |x|)$. What discontinuities has this function?

13. The function $y = \operatorname{sgn} x$, read "sign x," has the value -1, 0, or 1 according as x is negative, 0, or positive. Draw its graph. What discontinuities does it have?

14. Given that $f(x) = -x$ when $x < 0$, $f(x) = 2x + 1$ when $x \geqq 0$. Show that $f(x)$ is discontinuous at $x = 0$. Find the graph of $y = f(x)$.

15. Show that y is defined as a function of x by the statement: "y is 0 or 1 according as x is rational or irrational." Show also that this function is discontinuous for all values of x. Can its graph be represented?

16. Illustrate the fact that $f(x) = 1/x$ is discontinuous at $x = 0$ by finding a point between 0 and .1 where $f(x) - f(.1) > 10^5$.

17. Illustrate that $f(x) = x^2 + 3x$ is continuous at $x = 0$ by finding an interval (x', x'') in which $|f(x)| < .01$.

18. Find the limiting value of $\left(x + \frac{1}{x}\right) \Big/ \left(2x - \frac{1}{x}\right)$ when $x \to \infty$; also when $x \to 0$.

19. Show that $y^2 - x = 0$ defines y as a "two-valued function" of x in the x-interval $(0, \infty)$. What two one-valued functions $y = f(x)$ express this same relation of y to x?

20. Given that (1) u increases and v decreases; (2) u remains $< v$; (3) $v - u \to 0$; prove by the theorems of § 5 that u and v approach one and the same limit.

21. Construct geometrically (using the straight line and circle only) the points in Fig. 1 which correspond to the numbers 10/3, $-5/6$, $\sqrt{2}$, $\sqrt{3}$. The points corresponding to $\sqrt[3]{2}$ and π cannot be so constructed, but we assume that they exist. It will be proved later that to every point A on the line corresponds a real number a in the sense explained in § 1; but it is an assumption, a geometric postulate, that to every real number a corresponds a point A.

II. THE DERIVATIVE

22. The differential calculus. The differential calculus is concerned first of all with the problem: A variable y being defined as a function of another variable x by a given equation of the form $y = f(x)$, it is required to find how y varies when x varies continuously. The problem is solved by aid of the function defined in the following section.

23. The derivative. 1. First consider the function $y=x^2$. Let x_1 denote any given value of x, and y_1 the corresponding value of y, so that

$$y_1 = x_1^2 \tag{1}$$

Again let Δx denote any change[1] in the value of x, and Δy the corresponding change in the value of y. We call Δx, Δy corresponding *increments* of x, y. They may be positive or negative.

Since $y = x^2$, if the increment Δx be added to x_1, an increment Δy will be added to y_1 such that

$$y_1 + \Delta y = (x_1 + \Delta x)^2 \tag{2}$$

Subtract (1) from (2) and simplify the result; we find

$$\Delta y = 2\,x_1 \Delta x + (\Delta x)^2 \tag{3}$$

Divide both members of (3) by Δx; we get

$$\frac{\Delta y}{\Delta x} = 2\,x_1 + \Delta x \tag{4}$$

The right member of (4) shows that if we treat Δx as a variable and cause it to approach 0 as limit, the ratio $\Delta y/\Delta x$ will approach a definite limit, namely the number represented

[1] The letter Δ, read "delta," has no numerical meaning in this connection. It is merely part of the symbol for an increment.

by $2x_1$. This number $2x_1$ is called the *derivative of* $y = x^2$ *at the point* $x = x_1$. Thus the derivatives at the points $x = -1, 0, 3$ are $-2, 0, 6$.

The numbers $2x_1$ are values of the function $2x$. This function $2x$ is called the *derivative of the function* x^2. It may be obtained directly from $y = x^2$ by using x instead of x_1 to denote the initial value of x.

2. Let $y = f(x)$ be any given function and let x denote any value of x for which $f(x)$ is continuous, § 16. If the increment Δx be added to x, an increment Δy will be added to y such that

$$y = f(x) \quad \text{becomes} \quad y + \Delta y = f(x + \Delta x)$$

Subtract the first equation from the second; we get

$$\Delta y = f(x + \Delta x) - f(x) \tag{5}$$

By hypothesis, $f(x)$ is continuous for the value of x under consideration; hence, § 16, $\Delta y \to 0$ when $\Delta x \to 0$.

Divide both members of (5) by Δx; we obtain

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{6}$$

Here $[f(x + \Delta x) - f(x)]/\Delta x$ represents a known expression in x and Δx and, in the cases that we shall meet, when $\Delta x \to 0$ this expression will approach a definite expression in x as limit. This last expression, or function, is called the *derivative of* $y = f(x)$ and is denoted by the symbol $f'(x)$. And $[f(x + \Delta x) - f(x)]/\Delta x \to f'(x)$ means that for every value c of x for which $[f(c + \Delta x) - f(c)]/\Delta x$ approaches a definite number as limit when $\Delta x \to 0$, that number is $f'(c)$. Therefore, by definition,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{7}$$

The derivative of a function $f(x)$ *is the limit approached by the ratio of the increment of* $f(x)$ *to the increment of* x *when the increment of* x *approaches the limit* 0.

EXAMPLE 1. If $f(x) = x^3$, find $f'(x)$.

$$f(x + \Delta x) - f(x) = (x + \Delta x)^3 - x^3$$
$$= 3\,x^2\Delta x + 3\,x(\Delta x)^2 + (\Delta x)^3$$

Hence $\dfrac{f(x + \Delta x) - f(x)}{\Delta x} = 3\,x^2 + 3\,x\Delta x + (\Delta x)^2$

Therefore $$f'(x) = \lim_{\Delta x \to 0} (3\,x^2 + 3\,x\Delta x + (\Delta x)^2) = 3\,x^2.$$

EXAMPLE 2. Find the derivative of x^4; also the derivative of $3\,x^2 + 2\,x$ at $x = -1$.

24. Differentiation. The process of finding the derivative of a function $y = f(x)$ is called *differentiation.* It may be indicated by writing D_x before y or $f(x)$: thus, D_xy, read "$D\ x$ of y." We may also use D_xy as a symbol for $f'(x)$.

EXAMPLE 1. Differentiate $y = 1/x$.

Since $y = 1/x$, we have $y + \Delta y = 1/(x + \Delta x)$ and therefore

$$\Delta y = \frac{1}{x + \Delta x} - \frac{1}{x} = -\frac{\Delta x}{x(x + \Delta x)}$$

Hence $$\frac{\Delta y}{\Delta x} = -\frac{1}{x(x + \Delta x)}$$

and therefore (§ 7) $$D_xy = -\lim_{\Delta x \to 0} \frac{1}{x(x + \Delta x)} = -\frac{1}{x^2}$$

The reasoning holds good for all values of x except 0.

EXAMPLE 2. Differentiate $y = \sqrt{x}$.

Here $$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$$

When $\Delta x \to 0$ this expression approaches the form $0/0$. But, if both numerator and denominator be multiplied by $\sqrt{x + \Delta x} + \sqrt{x}$, the numerator becomes Δx, and, canceling this with the Δx in the denominator, we have

$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}$$

Hence (§ 8) $$D_xy = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

The reasoning holds good for all positive values of x and $x + \Delta x$. For $x = 0$ we have $\Delta y/\Delta x = 1/\sqrt{\Delta x}$, and this $\to \infty$ when $\Delta x \to 0$.

EXAMPLE 3. Differentiate the following functions:

1. $\sqrt{1 - x}$ 2. $\dfrac{1}{\sqrt{x}}$ 3. $\dfrac{2\,x - 1}{3\,x + 1}$ 4. $\dfrac{1}{x^2}$ 5. $\dfrac{x^2 + 1}{x + 1}$

25. On the existence of a derivative. 1. At a point $x = c$ where the ratio $[f(c + \Delta x) - f(c)]/\Delta x$ does not approach a finite limit when $\Delta x \to 0$, there is, properly speaking, no derivative. Nevertheless, in case $\lim\limits_{x \to c} |f'(x)| = \infty$, it is customary to say that $f(x)$ has an *infinite derivative* at $x = c$.

EXAMPLE 1. Both $1/x$ and its derivative $-1/x^2$ are infinite at $x = 0$.

2. *A function $f(x)$ cannot have a finite derivative at $x = c$ unless it is continuous at $x = c$: i.e.* unless $f(c + \Delta x) - f(c) \to 0$ when $\Delta x \to 0$.

On the other hand, $f(x)$ may be continuous at $x = c$ and yet not have a finite derivative at $x = c$.

EXAMPLE 2. At $x = 0$, $\sqrt{x}$ is continuous but its derivative $1/2\sqrt{x}$ is ∞.

EXAMPLE 3. The function $f(x) = x \sin(1/x)$ is continuous at $x = 0$ if we make $f(0) = \lim\limits_{x \to 0} f(x) = 0$. But $[f(0 + \Delta x) - f(0)]/\Delta x = \sin(1/\Delta x)$ does not approach a limit when $\Delta x \to 0$. Hence $x \sin(1/x)$ has no derivative at $x = 0$.

EXAMPLE 4. The function $y = |x|$, whose graph consists of the half-lines OA and OB, is continuous, but does not have a derivative at O; for

$$\frac{DP}{OD} = -1 \qquad \therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -1$$

$$\frac{EQ}{OE} = 1 \qquad \therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 1.$$

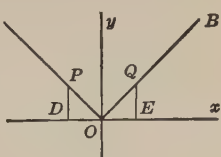

FIG. 7.

It may be said, however, that at O the function has a *left-hand derivative*, -1, and a *right-hand derivative*, 1.

26. Derivative of a constant. A constant c does not change when x is given the increment Δx. Hence $\Delta c = 0$ $\therefore \Delta c/\Delta x = 0$ $\quad \therefore \lim\limits_{\Delta x \to 0} \Delta c/\Delta x = 0$, that is,

$$D_x c = 0 \tag{1}$$

In like manner, if y be a variable *which is independent of* x, then $D_x y = 0$. For Δy denotes the increment which y receives as a consequence of giving x the increment Δx, and, if

y be independent of x, its value is not affected by a change in the value of x. Hence $\Delta y = 0$ and therefore $\lim\limits_{\Delta x \to 0} \Delta y/\Delta x = 0$. Thus Δx itself is a variable which is independent of x. Hence

$$D_x(\Delta x) = 0 \tag{2}$$

27. Derivative of a sum. Let $y = u + v$, where u, v denote functions of x which have finite derivatives for the values of x under consideration. When x is given the increment Δx, the functions u, v, y receive increments Δu, Δv, Δy such that $\Delta y = \Delta u + \Delta v$ and therefore

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

from which it follows, § 7, on making $\Delta x \to 0$, that

$$D_x y = D_x u + D_x v \tag{3}$$

The same reasoning applies to $u - v$, $u \pm v \pm w$, and so on. Hence

The derivative of the algebraic sum of any finite number of functions is the algebraic sum of their derivatives.

28. Derivative of cu. When x is given the increment Δx, and therefore u the increment Δu, then cu and u/c take the increments $c\Delta u$ and $\Delta u/c$. Dividing $c\Delta u$, $\Delta u/c$ by Δx and making $\Delta x \to 0$, we get

$$D_x cu = cD_x u \qquad D_x(u/c) = D_x u/c \tag{4}$$

To find the derivative of cu or u/c, replace u by $D_x u$.

29. Derivative of x^n. Let $y = x^n$, where n is a positive integer. When x is given the increment Δx, y takes an increment Δy such that

$$\Delta y = (x + \Delta x)^n - x^n$$

Set $x + \Delta x = x'$ and therefore $\Delta x = x' - x$. Then

$$\frac{\Delta y}{\Delta x} = \frac{x'^n - x^n}{x' - x} = x'^{n-1} + x'^{n-2}x + \cdots + x^{n-1}$$

There are n terms in this polynomial, and when $\Delta x \to 0$ and therefore $x' \to x$, each of these terms $\to x^{n-1}$. Hence

$$D_x x^n = nx^{n-1} \tag{5}$$

Thus $D_x x = 1$, $D_x x^2 = 2x$, $D_x x^3 = 3x^2$, $D_x x^4 = 4x^3$, and so on.

EXAMPLE 1. By the formulas (1), (3), (4), (5), we have
$D_x 5x^7 = 5 \cdot 7x^6 = 35x^6$, $D_x x^5/10 = 5x^4/10 = x^4/2$,
$D_x(2x^8 + 4x^4 - 7) = D_x 2x^8 + D_x 4x^4 - D_x 7 = 16x^7 + 16x^3$.

EXAMPLE 2. If $f(x) = (x - a)(x - 3a)$, find $f'(x)$.
$f(x) = x^2 - 4ax + 3a^2$. Hence $f'(x) = 2x - 4a$.

EXAMPLE 3. Differentiate each of the following with respect to x.

1. $4 - 3x + 5x^2 - 7x^3$
2. $2x^{12} - 10x^{11} + x^4 + 35$
3. $x^3 - cx^2 + c^2x - c^3$
4. $3(x - 2)(x + 4)x^2$
5. $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n$
6. $(x - 1)(x - 2)(x - 3)$
7. $x(1 - x)(1 + x) + x^2 + 5$

30. Slope property of the derivative. Let P and Q be the points (x_1, y_1) and $(x_1 + \Delta x, y_1 + \Delta y)$ of the graph of $y = f(x)$.

Since $PD = \Delta x$ and $DQ = \Delta y$, we have

$$\frac{\Delta y}{\Delta x} = \tan DPQ = \tan xSP$$

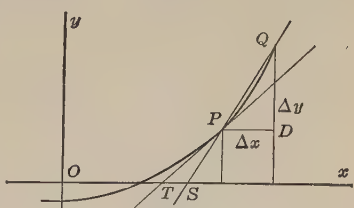

FIG. 8.

When $\Delta x \to 0$, Q moves along the curve toward P as limit and the line QPS turns about P toward a limiting position PT such that

$$\tan xTP = \lim_{\Delta x \to 0} \tan xSP = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_1) \tag{1}$$

By definition, PT is the tangent to the curve at P. Hence

The derivative of $y = f(x)$ at $x = x_1$, is the slope of the tangent to the graph at the point $[x_1, f(x_1)]$.

The slope of the tangent to a curve at any point is also called the *slope of the curve* at that point. When $f'(x)$ is con-

tinuous, the slope changes gradually from point to point and the curve is *smooth.*

The equation of the line through the point (x_1, y_1) and having the slope m is $y - y_1 = m(x - x_1)$; hence the *equation of the tangent* to the curve $y = f(x)$ at the point (x_1, y_1) is

$$y - y_1 = f'(x_1)(x - x_1) \tag{2}$$

EXAMPLE 1. Find the tangent and normal to the curve $y = x^3$ at the point whose x is 2. The point is (2, 8). Since $f'(x) = 3x^2$, we have $f'(2) = 12$. Hence the equation of the tangent is $y - 8 = 12(x - 2)$, or $12x - y - 16 = 0$. The normal, being perpendicular to the tangent, has the slope $-1/12$. Hence the equation of the normal is $y - 8 = (-1/12)(x - 2)$, or $x + 12y - 98 = 0$.

EXAMPLE 2. Find the points of the curve $y = x^3$ where the slope is 3.

Since $f'(x) = 3x^2 = 3$, we have $x = \pm 1$. Hence the points are (1, 1) and $(-1, -1)$.

EXAMPLE 3. At what point of the parabola $y = x^2 - 3x$ is the slope 0?

EXAMPLE 4. Find the tangent and normal to $y = x^3 - 12x$ at the point whose x is -1; also at the origin. What are the equations of the tangents parallel to Ox?

EXAMPLE 5. For what values of x is the slope of $y = 2x^3 - 9x^2 + 12x$ zero, positive, negative? At what points is the slope 12?

31. Sign of the derivative. Consider the values of $y = f(x)$ in the order of increasing values of x. Let x_1 be any value of x such that $f'(x_1)$ is finite and not 0, and let y_1 be the corresponding value of y;

If $f'(x_1)$ is positive, the value y_1 of y is immediately preceded by lesser values only and followed by greater values only.

If $f'(x_1)$ is negative, the value y_1 of y is immediately preceded by greater values only and followed by lesser values only.

For when $|\Delta x|$ is sufficiently small, $\Delta y/\Delta x$ has the same sign [1] as its limit $f'(x_1)$, therefore Δy the same sign as $f'(x_1)\Delta x$.

[1] Suppose that $u \to c$, where $c \neq 0$. Then ultimately $|u - c|$ remains $< |c|$. But $u = c + (u - c)$, and therefore when $|u - c| < |c|$ the sign of u is that of c.

Hence

If $f'(x_1)$ is $+$, then according as $\Delta x \lesseqgtr 0$ we have $\Delta y \lesseqgtr 0$ and therefore $y_1 + \Delta y \lesseqgtr y_1$.

If $f'(x_1)$ is $-$, then according as $\Delta x \lesseqgtr 0$ we have $\Delta y \gtreqless 0$ and therefore $y_1 + \Delta y \gtreqless y_1$.

32. Theorem. *If $f'(x)$ is positive throughout the x-interval (a, b), then $f(x)$ continually increases as x increases from a to b. If $f'(x)$ is negative throughout (a, b), $f(x)$ continually decreases as x increases from a to b.*

For let x_1 and x_2 be any two values of x such that $a \leqq x_1 < x_2 \leqq b$. By § 25, 2., $f(x)$ is continuous in (a, b); hence it has a greatest value M and a least value m in the interval (x_1, x_2), § 18, 1.

If $f'(x)$ is positive throughout (a, b), then $f(x_2)$ is M; for every other value of $f(x)$ in (x_1, x_2) is followed by greater values of $f(x)$ in (x_1, x_2), § 31. In like manner, $f(x_1)$ is m. Hence $f(x_1) < f(x_2)$.

Similarly, if $f'(x)$ is negative throughout (a, b), then $f(x_1) > f(x_2)$.

Example 1. When does $y = x^3 - 3x$ increase and when decrease as x increases?

$f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$

In interval $(-\infty, -1)$, $f'(x)$ is $+$

$\therefore$ y increases $(-\infty$ to $2)$

In interval $(-1, 1)$, $f'(x)$ is $-$

$\therefore$ y decreases $(2$ to $-2)$

In interval $(1, \infty)$, $f'(x)$ is $+$

$\therefore$ y increases $(-2$ to $\infty)$

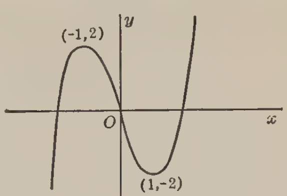

Fig. 9.

The graph rises from an infinite distance below Ox to the "maximum point" $(-1, 2)$, then falls to the "minimum point" $(1, -2)$, then rises to an infinite distance above Ox (§ 11, Ex. 1).

Example 2. Deal similarly with the following, drawing their graphs.

1. $y = x^3$ 2. $y = x^2 - 8x + 15$ 3. $y = x - x^2$ 4. $y = x^2(x-1)^2$.

33. Approximate values of Δy. Suppose $f'(x_1) \neq 0$. When $|\Delta x|$ is sufficiently small, $\Delta y / \Delta x$ differs but little numerically from its limit $f'(x_1)$, and therefore Δy but little from $f'(x_1)\Delta x$, that is, the difference is small as compared with $f'(x_1)\Delta x$. Hence, *for small values of* Δx

$f'(x_1)\Delta x$ is an approximate value of Δy

EXAMPLE 1. How much approximately does $y = x^2$ change when x changes from 20 to 20.1? $f'(x_1) = 2x_1 = 40$, $\Delta x = .1$ $\therefore \Delta y = 40(.1) = 4$ approximately. The exact value of Δy is 4.01.

EXAMPLE 2. Find the approximate change in $y = x^3 + 4x^2$ when x changes from 12 to 11.98.

EXAMPLE 3. If x start at 15, how much approximately may it change and change the value of $y = 2x^2 + 5x$ by not more than .2?

34. Differentials. 1. The product $f'(x)\Delta x$ is called the *differential* of $y = f(x)$ and is denoted by dy or $df(x)$. Hence, by definition,

$$dy = f'(x)\Delta x \tag{1}$$

When $y = x$, then $f'(x) = 1$, and (1) becomes $dx = \Delta x$. Hence the definition (1) implies that the differential of the *independent* variable is the same as its increment. We may therefore also write (1) in the form

$$dy = f'(x)dx \tag{2}$$

EXAMPLE. Thus $d(x^3) = 3x^2dx$, $d(2x^2 + 3x + 5) = (4x + 3)dx$.

2. The geometric meaning of dy is shown in Fig. 10; also its relation to Δy. P and Q are the points (x, y) and $(x + \Delta x, y + \Delta y)$, PE is the tangent to the curve $y = f(x)$ at P,

$$PD = \Delta x = dx \qquad DQ = \Delta y$$
$$DE = \tan DPE \cdot \Delta x = f'(x)dx = dy$$

3. Dividing both members of (2) by dx, we get

$$\frac{dy}{dx} = f'(x) \tag{3}$$

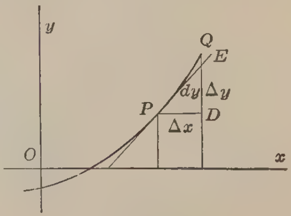

FIG. 10.

The new symbol dy/dx thus obtained for the derivative is the one most used. We shall ordinarily use it instead of D_xy. Thus the equation of the tangent at a curve point (x_1y_1), § 30(2), is written

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1) \tag{4}$$

4. Instead of D_x itself, we often use $\frac{d}{dx}$. Thus $\frac{d}{dx}x^2 = 2\,x$.

35. Rate of change of a function. 1. When y is so related to x that the ratio $\Delta y/\Delta x$ of any corresponding changes in their values has a constant value, r, then *y is said to vary uniformly with x*, and r is called the *rate of change of y with respect to x*. The numerical value of r is the number of units change in y per unit change in x and the sign of r indicates whether the rate is one of increase or decrease.

If x_1, y_1 denote any particular pair of corresponding values of x and y, and x, y themselves any other pair, we may set $\Delta x = x - x_1$, $\Delta y = y - y_1$ in the equation $\Delta y/\Delta x = r$, which gives

$$y - y_1 = r(x - x_1) \tag{1}$$

Therefore, if y varies uniformly with x, then y is connected with x by an equation of the first degree whose graph is a straight line having r for its slope. The converse is true; for, if $y = mx + c$, then $\Delta y/\Delta x = m$. Thus, if $y = 6\,x + 5$, then $\Delta y/\Delta x = 6 = r$.

2. When y is connected with x by an equation $y = f(x)$ which is not of the first degree, y does not vary uniformly with x.

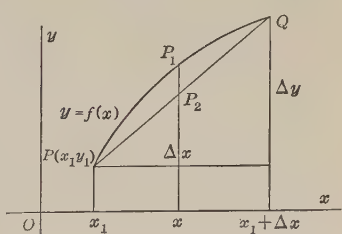

FIG. 11.

Let x_1, y_1 be corresponding values of x, y, and Δx, Δy corresponding increments, and suppose $f'(x_1)$ to exist and to be finite.

When x varies continuously from x_1 to $x_1 + \Delta x$, y varies continuously from y_1 to $y_1 + \Delta y$, the corresponding point $P_1(x, y)$ moving on the arc PQ of the graph of $y = f(x)$. But the result of this variation is the same as that effected by a uniform variation at the rate $\Delta y/\Delta x$ in which the corresponding point P_2 moves on the chord PQ; and when $\Delta x \rightarrow 0$, the interval $(x_1, x_1 + \Delta x)$ closes in on the point x_1, the arc PQ approaches coincidence [1] with the chord PQ, and the rate $\Delta y/\Delta x$ approaches the limit $f'(x_1)$. This limit is therefore called the rate of change of $y = f(x)$ with respect to x at the point $x = x_1$: that is, by definition,

The rate of change of $y = f(x)$ with respect to x at $x = x_1$ is

$$\frac{dy_1}{dx_1} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_1) \qquad (2)$$

The ratio $\Delta y/\Delta x$ is called the *mean rate* of change of $y = f(x)$ with respect to x in the interval $(x_1\, x_1 + \Delta x)$. Hence the rate at $x = x_1$ may be defined as the limit approached by this mean rate as $\Delta x \rightarrow 0$.

Observe that the rate at $x = x_1$ equals the slope of the graph at $x = x_1$.

Example 1. If $y = x^2$, then $dy/dx = 2\,x$. Hence at $x = 3$, the rate of change of y with respect to x is 6, a rate for which y would increase 6 units when x increased 1 unit.

Example 2. If x be the length in inches of an edge of a cube and V its volume (in cu. in.), then $V = x^3 \quad \therefore dV/dx = 3\,x^2$. Hence, when $x = 5$, the rate of change of V with respect to x is 75 cu. in./in.

Example 3. For what values of x is the rate of change of $x^3 - 3\,x^2$ with respect to x equal to 9?

Example 4. The current in an electric circuit varies inversely as the resistance. If $c = 40$ when $R = 6$, find the rate of change of c with respect to R when $R = 12$.

Example 5. Find a formula for the rate of change of the area A of a circle with respect to its radius r; also for the rate of change of r with respect to A.

[1] By this is meant that $P_2P_1/\Delta x \rightarrow 0$ when $\Delta x \rightarrow 0$.

Example 6. Find the rate of change of the area of an equilateral triangle with respect to its altitude when the altitude is 8.

36. Rectilinear motion. Suppose that a point P is moving in a straight line. Let s denote its distance, in feet, from a fixed point O on the line t seconds from some given instant, the sign of s being $+$ or $-$ according as the direction from O to P is that taken as positive or negative on the line.

1. If P traverses equal spaces in equal intervals of time, its motion is said to be *uniform*. For in this case s varies uniformly with t, the ratio $\Delta s/\Delta t$ of corresponding changes in s and t being constant. This constant ratio — the rate of change of s with respect to t — is called the *velocity* of the motion (in feet per second) and is denoted by v.

The equation connecting s and t has the form $s = vt + s_0$, where s_0 is the distance of P from O when $t = 0$.

2. If the motion of P is not uniform but yet is in accordance with some known law expressed by a given equation $s = f(t)$, we define its velocity at any given instant t_1 as the rate of change of $s = f(t)$ with respect to t at t_1, that is, as the value at t_1 of

$$v = \frac{ds}{dt} \tag{1}$$

The sign of v at t_1 indicates the direction of the motion at t_1; for s is increasing when ds/dt is $+$, decreasing when ds/dt is $-$, § 31.

The numerical value of v is called the *speed*.

3. The rate of change of v with respect to t at any instant t_1 is called the *acceleration* at t_1. It is the value at t_1 of the variable or constant α defined by the formula:

$$\alpha = \frac{dv}{dt} \tag{2}$$

When α is $+$, the velocity is increasing; when $-$, decreasing.

EXAMPLE 1. The distance s in feet that a body dropped from rest will fall in t seconds, the resistance of the air being disregarded, is approximately

$$s = 16\,t^2$$

Since $v = ds/dt = 32\,t$, the velocity at the end of the 1st, 2nd $\cdots$ second is 32, 64 $\cdots$ ft./sec. Since $\alpha = dv/dt = 32$, the acceleration is constant: 32 ft./sec^2.

EXAMPLE 2. A point P moves in a straight line. At the end of t seconds its distance from O is $s = t^3 - 7\,t^2 + 8\,t$ feet. Discuss its motion.

Here $v = 3\,t^2 - 14\,t + 8 = (3\,t - 2)(t - 4)$, which is $+$ in the t-interval $(0, 2/3)$, $-$ in the interval $(2/3, 4)$, and $+$ when $t > 4$. Hence P first moves forward for 2/3 of a second, then backward for $3\frac{1}{3}$ seconds, then forward again.

Also $\alpha = 6\,t - 14$, which is $-$ for $t < 7/3$ and $+$ for $t > 7/3$.

Hence the velocity first decreases for 7/3 seconds, then increases.

EXAMPLE 3. Let P be moving along a straight line, and let s denote its distance from O at the time t (before or after the instant of reference according as the sign of t is $-$ or $+$). In each of the following cases discuss the motion of P as in Ex. 2; also find the velocity and acceleration at the time indicated.

1. $s = 2\,t^3 - 21\,t^2 + 60\,t - 200,\ t = 3$ **2.** $s = t^4 - 2\,t^2,\ t = -5.$

EXAMPLE 4. If a projectile P is shot vertically upward from a platform s_0 feet from the ground, with an initial speed v_0 ft./sec., at the end of t seconds it will be $s = s_0 + v_0t - 16\,t^2$ feet above the ground. If $s_0 = 20$, $v_0 = 1024$, find the two instants at which P is 14100 ft. above the ground and its velocities at these instants.

EXERCISE II

1. Find the derivative and the differential of each of the following:

(1) $3\,x^4 - x^3/6 + (x^2 - 8\,x)/2 + 4$ (2) $x(x - 1)(x^2 + x + 1)$

2. When does $y = x^2$ decrease and when increase? Draw the graph carefully, first plotting the points where $x = 0, \pm 1/2, \pm 1, \pm 2, \pm 3$. Find the tangent and normal where $x = 1$. Find the angles which it makes[1] with the line $y - x - 2 = 0$.

3. When does $y = x^3 - 3\,x^2 - 9\,x$ increase and when decrease? Find its graph. Find the tangent and normal where $x = 1$. Find the points where the slope is -9.

[1] If two curves C_1 and C_2 intersect at the point P and their slopes at P are m_1 and m_2, the angle θ between C_1 and C_2 at P is given by the formula $\tan\theta = (m_1 - m_2)/(1 + m_1m_2)$.

4. The sides of a rectangle are $x + 2$ and $2x + 3$. At what rate with respect to x is its area increasing when x increasing passes through the value 5?

5. What is the change approximately in this area (Ex. 4) when x increases from 5 to 5.25?

6. Approximately how much does $y = x^3 - 2x^2 + 5$ change when x changes from 10 to 9.96?

7. By how much approximately may x, starting at 7, increase and $y = x^4 + x^2$ change not more than .3?

8. The edge of a cubical box is supposed to be a foot long. How much approximately may this be in error and the actual volume differ from 1 cu. ft. by not more than 1 cu. in.?

9. Find approximate formulas for the area of a narrow circular ring and the volume of a thin spherical shell.

10. If $y = x^3 + 3x^2$, find $\Delta y - dy$ in terms of x and Δx; also its value when $x = 10$ and $\Delta x = .1$.

11. A soap bubble is being blown; at what rates with respect to its radius r are its surface and volume increasing when $r = 3$?

12. The equation of motion of a ball rolling down a certain inclined plane is $s = 5t^2$ (in feet and seconds). Find v when $t = 4$; t and s when $v = 10$; and α.

13. A ball which starts rolling up a certain incline is at the end of t seconds at the distance (in feet) $s = 3t - t^2$ from the starting point. What is its initial velocity? When does it begin rolling downward?

14. The equations of motion of two points P_1 and P_2 on the same line are $s_1 = 2t^2 + 2$, $s_2 = t^2 + t + 1$. Show that P_1 and P_2 are nearest each other when $t = 1/2$.

15. Show that if $s = 16t^2$, then $v = 8\sqrt{s}$. What is the rate of change of v with respect to s when $s = 8$? when $s = 20$? See § 24, Ex. 2.

16. Let $y = f(x)$ be a function whose rate of change $f'(x)$ is continuous and increases (or decreases) throughout the interval $(x_1, x_1 + \Delta x)$, Fig. 11. Show by § 18, 2 that there is a point x' in $(x_1, x_1 + \Delta x)$ such that $f'(x') = \Delta y/\Delta x$ and therefore $\Delta y = f'(x')\Delta x$.

III. DERIVATIVES OF ALGEBRAIC FUNCTIONS

37. Derivative of a product. 1. Let $y = uv$, where u and v denote functions of x which have finite derivatives for the values of x under consideration. Give x the increment Δx. Then u, v, y receive increments Δu, Δv, Δy such that

$$y = uv \qquad \text{becomes} \qquad y + \Delta y = (u + \Delta u)(v + \Delta v)$$

Subtract the first equation from the second and divide by Δx.

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x}(v + \Delta v) + u\frac{\Delta v}{\Delta x}$$

Let $\Delta x \rightarrow 0$. Then $v + \Delta v \rightarrow v$, § 25, 2., and, by § 7, we have

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} \tag{1}$$

2. Let $y = uvw$. By (1), we have

$$\begin{aligned}\frac{d}{dx}(uvw) &= \frac{du}{dx} \cdot vw + \frac{d}{dx}(vw) \cdot u \\ &= \frac{du}{dx}vw + \frac{dv}{dx}wu + \frac{dw}{dx}uv \qquad (2)\end{aligned}$$

A repetition of this process leads to the rule:

To differentiate the product of two or more functions, multiply the derivative of each function by the product of the other functions and add the products thus obtained.

38. Derivative of a quotient. Let $y = u/v$, where $v \neq 0$. We have

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}$$

$$\therefore \qquad \frac{\Delta y}{\Delta x} = \left[v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}\right]/v(v + \Delta v)$$

Let $\Delta x \to 0$. By § 7, we have

$$\frac{d}{dx}\frac{u}{v} = \left[v\frac{du}{dx} - u\frac{dv}{dx}\right]/v^2 \tag{3}$$

The derivative of a fraction is denominator times derivative of numerator, minus numerator times derivative of denominator, all divided by square of denominator.

EXAMPLE 1. $\frac{d}{dx}(x-a)(x-b)(x-c) = \frac{d}{dx}(x-a)\cdot(x-b)(x-c)$

$$+\frac{d}{dx}(x-b)\cdot(x-c)(x-a)+\frac{d}{dx}(x-c)\cdot(x-a)(x-b)$$

$$=(x-b)(x-c)+(x-c)(x-a)+(x-a)(x-b).$$

EXAMPLE 2. $\frac{d}{dx}\frac{x^2}{3x+5} = \frac{(3x+5)2x - x^2\cdot 3}{(3x+5)^2} = \frac{3x^2+10x}{(3x+5)^2}$

EXAMPLE 3. Differentiate each of the following with respect to x.

1. $(ax+b)(cx+d)$ **2.** $(x^2+a^2)(x^2+b^2)$ **3.** $(x^2+c^2)/x^2$

4. $\frac{2x^2-8x+5}{x^3-10}$ **5.** $\frac{(x-1)(2-3x)}{x^2(3-x)}$ **6.** $\frac{ax+b}{cx+d}$

7. $(x^2-3x+4)(x^2+2x-1)$ **8.** $(2x+3)(x-4)(5-2x)$

39. Derivative of x^n. It is true for *all rational values of* n that

$$\frac{d}{dx}x^n = nx^{n-1} \tag{1}$$

1. Let $y = x^{p/q}$, where p, q are positive integers. We have

$$\Delta y = (x+\Delta x)^{p/q} - x^{p/q}$$

Set $x^{1/q} = z$ and $(x+\Delta x)^{1/q} = z'$

Then $x = z^q$ and $x+\Delta x = z'^q$ $\therefore \Delta x = z'^q - z^q$

Also $x^{p/q} = z^p$ and $(x+\Delta x)^{p/q} = z'^p$ $\therefore \Delta y = z'^p - z^p$

Hence $\frac{\Delta y}{\Delta x} = \frac{z'^p - z^p}{z'^q - z^q} = \frac{z'^{p-1} + z'^{p-2}z + \cdots + z^{p-1}}{z'^{q-1} + z'^{q-2}z + \cdots + z^{q-1}}$

When $\Delta x \to 0$, each of the p terms in the numerator $\to z^{p-1}$ and each of the q terms in the denominator $\to z^{q-1}$, § 8. Hence

$$\frac{d}{dx}x^{p/q} = \frac{pz^{p-1}}{qz^{q-1}} = \frac{p}{q}z^{p-q} = \frac{p}{q}(x^{1/q})^{p-q} = \frac{p}{q}x^{p/q-1} \tag{a}$$

2. Let s denote any positive rational number. By (a) and § 38 (3)

$$\frac{d}{dx}x^{-s} = \frac{d}{dx}\frac{1}{x^s} = -\frac{sx^{s-1}}{x^{2s}} = (-s)x^{-s-1} \tag{b}$$

Both (a) and (b) are of the form (1), which proves the theorem.

EXAMPLE 1. $\frac{d}{dx}x^{1/3} = \frac{1}{3}x^{-2/3}$, $\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$, $\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = -x^{-2}$, $\frac{d}{dx}\frac{x^2+1}{x^{1/2}} = \frac{d}{dx}(x^{3/2}+x^{-1/2}) = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-3/2}$.

EXAMPLE 2. Differentiate the following:

$\sqrt[3]{x^5}$, x^{-7}, $5/x^4$, $2/5\,x$, $(2\,x^2 - 3\,x^{1/2} + 1)/2\,x$.

40. Derivative of a function of a function. 1. Let y be a function of a variable u which is itself a function of x: that is, let

$$y = f(u) \qquad \text{where} \qquad u = \phi(x)$$

and suppose that, for all values of x under consideration and the corresponding values of u, the derivatives $D_u y = f'(u)$, $D_x u = \phi'(x)$ exist and are finite. Then

$$D_x y = D_u y\, D_x u = f'(u)\phi'(x) \tag{1}$$

The derivative of y with respect to x is the derivative of y with respect to u times the derivative of u with respect to x.

For give x the increment Δx and let Δu and Δy be the corresponding increments of $u = \phi(x)$ and $y = f(u)$. We have

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}$$

Let $\Delta x \to 0$. Then, by hypothesis, $\Delta u/\Delta x \to$ the finite limit $\phi'(x)$; therefore also $\Delta u \to 0$ and $\Delta y/\Delta u \to f'(u)$. Hence[1] $\Delta y/\Delta x \to f'(u)\phi'(x)$.

[1] It is not permissible to express $\Delta y/\Delta x$ in the form $[\Delta y/\Delta u][\Delta u/\Delta x]$ when Δu is 0. Hence the proof given above does not suffice for a function $u = \phi(x)$ and a value of x such that Δu is 0 for certain values of Δx as small as we please but $\neq 0$. But it can be shown that in such a case both $\phi'(x)$ and $D_x f[\phi(x)]$ are 0 at the point x in question, so that the formula (1) is true at that point. An example is $u = x^2 \sin(1/x)$ at $x = 0$.

2. If we multiply both members of (1) by dx we get

$$D_xy\,dx = f'(u)\phi'(x)dx$$

But, by § 34 (2), $D_xy\,dx = dy$ and $\phi'(x)dx = du$. Hence

$$dy = f'(u)du \tag{2}$$

By (2), we can replace $f'(u)$ in (1) by dy/du just as, by § 34 (3), we can replace D_xy by dy/dx and $\phi'(x)$ by du/dx. If this be done, (1) becomes the easily remembered identity

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{3}$$

3. Observe that (2) shows that if $y = f(u)$, then dy may be expressed in terms of du in the same manner, § 34 (2), as if u were the independent variable.

Example. If $y = u^3$ and $u = x^2 + x$, find dy/dx.

$$\frac{dy}{dx} = \frac{d}{du}u^3 \cdot \frac{d}{dx}(x^2 + x) = 3\,u^2(2\,x + 1) = 3(x^2 + x)^2(2\,x + 1)$$

The most frequent application of (1) or (3) is in differentiating a function given in the form $f(u)$ where u denotes some expression in x. Thus, § 39 (1),

$$\frac{d}{dx}u^n = \frac{du^n}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx} \tag{4}$$

Examples. $\dfrac{d}{dx}(3\,x - 5)^4 = \dfrac{d(3\,x - 5)^4}{d(3\,x - 5)}\dfrac{d(3\,x - 5)}{dx} = 4(3\,x - 5)^3 \cdot 3.$

$$\frac{d}{dx}\left(\frac{1 - x}{1 + x}\right)^3 = 3\left(\frac{1 - x}{1 + x}\right)^2 \cdot \frac{d}{dx}\left(\frac{1 - x}{1 + x}\right) = 3\left(\frac{1 - x}{1 + x}\right)^2 \frac{-2}{(1 + x)^2}.$$

$$\frac{d}{dx}\frac{a}{(bx + c)^2} = a\frac{d}{dx}(bx + c)^{-2} = -2\,a(bx + c)^{-3}b = -\frac{2\,ab}{(bx + c)^3}$$

$$\frac{d}{dx}\sqrt{1 - x^2} = \frac{d}{dx}(1 - x^2)^{1/2} = \frac{1}{2}(1 - x^2)^{-1/2}(-2\,x) = -\frac{x}{\sqrt{1 - x^2}}$$

EXERCISE III

Differentiate each of the following with respect to the variable of which it is a function. If the expression can be more easily differentiated in another form, first reduce it to that form.

1. $x^3(5-3x)^4$ 2. $(1+x)^3(1-2x)^4$ 3. $(1+x^2)^3(1-2x^2)^5$

4. $\dfrac{1+2t}{3-t}$ 5. $\left(\dfrac{1+2t}{3-t}\right)^6$ 6. $\dfrac{(2x^2-x)^2}{x^4+2}$

7. $x(1-x)^2(1+x)^3$ 8. $\sqrt[3]{1+2x}$ 9. $\sqrt{bx+c}$

10. $\sqrt[4]{x(1+x)}$ 11. $\sqrt{1+2t}\cdot\sqrt{1-3t}$ 12. $\sqrt{a^2-x^2}$

13. $\dfrac{z}{\sqrt{z^2+a^2}}$ 14. $\dfrac{1}{\sqrt{x}+\sqrt{x+2}}$ 15. $\left(\dfrac{1-x^2}{1+x^2}\right)^{1/2}$

16. $\dfrac{\sqrt{ax^2+b}}{\sqrt{cx^2+d}}$ 17. $\left(\dfrac{y^{3/2}}{3}+\dfrac{y^{5/2}}{5}\right)^{1/2}$ 18. $\dfrac{x^{2/3}+\sqrt[3]{x}-2x^{-1}}{x}$

19. $\dfrac{(1+2x)^2}{(1+3x)^4}$; also reduced to the form $(1+2x)^2(1+3x)^{-4}$.

20. $x\sqrt{x+5}+(x+5)^{3/2}$ 21. $(x+2)\sqrt{x^2+3x+2}$

22. Prove: If $y = u^{1/2}$ and $u = a^2 - x^2$, then

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=-\frac{x}{\sqrt{a^2-x^2}}$$

23. Prove: If $y = u^{1/2}$, $u = \dfrac{1-v}{1+v}$, $v = x^2$, then

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}=\frac{-2x}{(1+x^2)\sqrt{1-x^4}}.$$

24. If x is increasing at the rate of 10 ft./sec., at what rate with respect to the time t is the diagonal of the rectangle whose sides are x and $x+4$ increasing when $x = 12$ feet?

41. Derivatives of higher orders. The derivative of the derivative of $y = f(x)$ is called its *second derivative* and is denoted by $D^2{}_x y$ or $f''(x)$. Similarly the derivative of the second derivative is called the *third derivative*: $D^3{}_x y = f'''(x)$; and so on.

Thus, if $y = 2x^3 - 3x^2 + 5x - 7 = f(x)$

then

$$\begin{aligned} D_x y &= 6x^2 - 6x + 5 &&= f'(x)\\ D^2{}_x y &= 12x - 6 &&= f''(x)\\ D^3{}_x y &= 12 &&= f'''(x)\end{aligned}$$

and $D^4{}_x y$ and all subsequent derivatives are 0.

The differential of $dy = f'(x)dx$ is called the *second differential* of y and is denoted by d^2y; that is, § 34,

$$d^2y = D_x(dy)dx = D_x[f'(x)dx]dx$$

But since $dx = \Delta x$ is independent of x and therefore $D_x(dx)$ is 0, § 26, (2), we have $D_x[f'(x)dx] = f''(x)dx$. Hence

$$d^2y = [f''(x)dx]dx = f''(x)(dx)^2 \quad \text{and} \quad \frac{d^2y}{dx^2} = f''(x) \qquad (1)$$

where dx^2 stands for $(dx)^2$. Similarly, if $d^3y = D_x(d^2y)dx$, and so on, we get

$$\frac{d^3y}{dx^3} = D^3{}_xy,\ \frac{d^4y}{dx^4} = D^4{}_xy,\ \cdots,\ \frac{d^ny}{dx^n} = D^n{}_xy \qquad (2)$$

These differential symbols for the derivatives of higher order are the ones most frequently used.

Example 1. Find the 1st, 2nd, $\cdots$ nth derivatives of

$$y = 1/(x+2) = (x+2)^{-1}.$$

$$\frac{dy}{dx} = (-1)(x+2)^{-2},\ \frac{d^2y}{dx^2} = (-1)(-2)(x+2)^{-3},\ \cdots,$$

$$\frac{d^ny}{dx^n} = (-)^n n!\,(x+2)^{-(n+1)} = \frac{(-)^n n!}{(x+2)^{n+1}}$$

Example 2. Find the successive derivatives of $x^4 - 2x^3 + 5x^2 - x$.

Example 3. Find the second derivative of $\sqrt{1+x^2}$.

Example 4. If $y = 1/(1-x)$, find d^3y/dx^3, d^5y/dx^5, d^ny/dx^n.

Example 5. Find the nth derivative of

(a) $2/(3x+4)$, (b) $1/(x+3)^3$, (c) $(1+x)^{1/2}$.

Example 6. Show that, if the successive derivatives of u, v are u_1, u_2, $\cdots$; v_1, v_2, $\cdots$,

1. $D^2{}_xuv = u_2v + 2u_1v_1 + uv_2$
2. $D^3{}_xuv = u_3v + 3u_2v_1 + 3u_1v_2 + uv_3$

and so on, the coefficients being those in the expansion of $(a+b)^n$ by the Binomial Theorem.[1]

Example 7. Find 2nd and 3rd derivatives of $x(1-x)^{1/2}$ by setting $u = (1-x)^{1/2}$, $v = x$ in Ex. 6, 1., 2.

42. Differentiation of implicit algebraic functions. 1. Let $f(x, y)$ denote a polynomial in x, y, that is, a sum of terms of

[1] This formula for the nth derivative of the product of two functions is called Leibnitz's theorem.

the type cx^my^n, where m, n are positive integers or 0; then $f(x, y) = 0$ is called an *algebraic equation* in x, y.

Suppose that $f(x, y) = 0$ can be solved for y in terms of x, and let $y = \phi(x)$ denote one of the solutions; the function $y = \phi(x)$ is called an *algebraic function*, and it is said to be *defined implicitly* by $f(x, y) = 0$. It can be proved that such functions have derivatives.

The graph of $f(x, y) = 0$ consists of the graphs of all the functions $y = \phi(x)$.

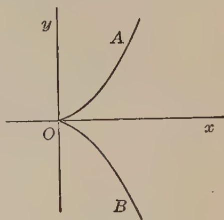

FIG. 12.

EXAMPLE 1. The equation $y^2 - x^3 = 0$ has the solutions $y = x^{3/2}$ and $y = -x^{3/2}$. Hence both $y = x^{3/2}$ and $y = -x^{3/2}$ are algebraic functions defined implicitly by $y^2 - x^3 = 0$. Both have derivatives. The graphs of $y = x^{3/2}$ and $y = -x^{3/2}$ are OA and OB. Together they form the graph AOB of $y^2 - x^3 = 0$.

2. Let $D_xf(x, y)$ denote the result of differentiating $f(x, y)$ term by term, regarding y as a function of x.

EXAMPLE 2. Thus, if

$$f(x, y) = y^2 + 2xy + 3x^2 + 4y - 6x + 11$$

then
$$\begin{aligned} D_xf(x, y) &= 2y\frac{dy}{dx} + 2\left(y + x\frac{dy}{dx}\right) + 6x + 4\frac{dy}{dx} - 6 \\ &= (2y + 6x - 6) + (2y + 2x + 4)\frac{dy}{dx} \end{aligned}$$

If $f(x, y) = 0$ is satisfied by $y = \phi(x)$, then $D_xf(x, y) = 0$ is satisfied by $y = \phi(x)$, $dy/dx = \phi'(x)$. For $f[x, \phi(x)] \equiv 0$ and therefore $D_xf[x, \phi(x)] \equiv 0$. Hence

If y denote a function of x defined by $f(x, y) = 0$, then corresponding values of x, y, dy/dx satisfy the equation $D_xf(x, y) = 0$.

3. As Ex. 2 shows, $D_xf(x, y) = 0$ can be reduced to the form $M + N(dy/dx) = 0$, where M, N involve x, y only. It therefore gives the slope dy/dx at any point P of the graph C of $f(x, y) = 0$ where $N \neq 0$. For this dy/dx is the slope

at P of the graph C_1 of one of the functions $y = \phi(x)$, and C_1 is a part of C.

EXAMPLE 3. Thus, differentiating $y^2 - x^3 = 0$ gives $2y\frac{dy}{dx} - 3x^2 = 0$. Hence at the point (1, 1) we have $2\frac{dy}{dx} - 3 = 0 \quad \therefore \frac{dy}{dx} = \frac{3}{2}$.

4. As Ex. 2 also shows, M may be got by differentiating $f(x, y)$ with respect to x, treating y as a constant.[1] It is therefore called the *partial derivative of* $f(x, y)$ *with respect to* x and is denoted by $\partial f/\partial x$. Similarly $N = \partial f/\partial y$. Hence the equation $D_x f(x, y) = 0$ may be written

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \tag{1}$$

and gives the slope dy/dx of C at any point P where $\partial f/\partial y \neq 0$.

It will be shown later that at a point P where $\partial f/\partial y = 0$ and $\partial f/\partial x \neq 0$, the slope of C is infinite.

A point P where $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ is called a *singular point* of C. We shall find that at such a point C may have one tangent, more than one, or none.[2] The point O in Fig. 12 is a singular point of the curve $y^2 - x^3 = 0$.

5. If u be given as a function of x by two equations of the form $u = F(x, y)$, $f(x, y) = 0$, and it be known that it has

[1] That this is true for any $f(x, y) = \Sigma c x^m y^n$ follows from the fact that

$$\frac{d}{dx}(x^m y^n) = m x^{m-1} y^n + x^m n y^{n-1}\frac{dy}{dx} = \frac{\partial}{\partial x}(x^m y^n) + \frac{\partial}{\partial y}(x^m y^n)\frac{dy}{dx}$$

[2] In deriving (1) we have supposed that P belongs to but one of the solutions $y = \phi(x)$ of $f(x, y) = 0$ and that $dy/dx = \phi'(x)$ is finite at P. This is true when $\partial f/\partial y \neq 0$ at P. But when $\partial f/\partial y = 0$, P belongs to more than one $y = \phi(x)$; the slopes of these $y = \phi(x)$'s at P are ∞ when $\partial f/\partial x \neq 0$, but they may be finite and distinct, equal, or imaginary when $\partial f/\partial x = 0$. These several possibilities are illustrated at the point O on the graphs of $y^2 - x = 0$, $y^2 - x^2 = 0$, $y^2 - x^3 = 0$, $y^2 + x^2 = 0$.

It may be added that, generally speaking, when the degree of $f(x, y)$ in y exceeds 4 the functions $y = \phi(x)$ cannot be found by solving $f(x, y) = 0$ *algebraically* — that is, by means of radicals — but in a sense and by methods which will be explained later.

an x-derivative when y has one, the method of implicit differentiation enables us to find du/dx in terms of x, y.

EXAMPLE 4. Given $u = (x^2 + y^2)^{1/2}$, and $xy - 2 = 0$, find du/dx.

Diff'g $\quad u^2 = x^2 + y^2 \quad$ gives $\quad u\dfrac{du}{dx} = x + y\dfrac{dy}{dx}$

Diff'g $\quad xy - 2 = 0 \quad$ gives $\quad y + x\dfrac{dy}{dx} = 0$

Eliminating dy/dx and solving for du/dx, $\quad \dfrac{du}{dx} = \dfrac{x^2 - y^2}{x(x^2 + y^2)^{1/2}}$

EXAMPLE 5. If $u = y/x$ and $y^2 - 3\,xy + 2\,x^2 - 4\,y = 0$, find du/dx in terms of x, y.

43. Tangents. In the equation $y - y_1 = \dfrac{dy_1}{dx_1}(x - x_1)$ replace $\dfrac{dy_1}{dx_1}$ by $-\dfrac{\partial f}{\partial x_1}\Big/\dfrac{\partial f}{\partial y_1}$, § 42 (1), and simplify. We obtain

$$(x - x_1)\frac{\partial f}{\partial x_1} + (y - y_1)\frac{\partial f}{\partial y_1} = 0 \tag{1}$$

EXAMPLE 1. After showing that the point (1, 2) is on the curve $y^2 - 2\,xy + 2\,x^2 - 2\,x = 0$, find the equation of the tangent at this point.

At (1, 2) $\quad \dfrac{\partial f}{\partial x} = -2\,y + 4\,x - 2 = -2 \qquad \dfrac{\partial f}{\partial y} = 2\,y - 2\,x = 2$

Hence the tangent is $-2(x - 1) + 2(y - 2) = 0$, or $x - y + 1 = 0$.

EXAMPLE 2. At what points of the circle $x^2 + y^2 = 5$ is the slope -2?

Diff'g $\quad x^2 + y^2 = 5$ and simplifying gives $\quad x + y\dfrac{dy}{dx} = 0$

At the required points $dy/dx = -2$. Hence these points satisfy $x - 2\,y = 0$ and $x^2 + y^2 = 5$ and therefore are (2, 1) and (− 2, − 1).

EXERCISE IV

1. Find the tangents and normals to the following curves:

1. $x^3 + x^2y + y^3 = 9$ at $(-1, 2)$
2. $y^2 + 3\,xy + 2\,x - 3\,y = 0$ at $(0, 0)$
3. $2\,x^2y - 3\,y^2 - 2\,x + 7 = 0$ at $(2, 3)$
4. $x^2 + xy + y^2 - 1 = 0$ at $(0, 1)$

2. Find the tangents to $3x^2 - 2xy + y^2 - 3y = 0$ at the points whose abscissa is 2.

3. Find the points of $2x^2 + y^2 - 4x = 0$ where the tangents are parallel to $x + y = 0$.

4. Find the points of $5x^2 - 2xy + 2y^2 = 45$ where the slope is 0; also the points where the slope is ∞. What are the equations of the tangents at these points?

5. Show that the hyperbolas $y^2 - x^2 = c$ and $xy = c'$ cut at right angles.

6. Show that the tangent at the point (x_1, y_1) of the hyperbola $2xy = a$ is $xy_1 + yx_1 = a$, and that the area of the triangle bounded by Ox, Oy and the tangent is a.

7. A point P is moving on $y^2 - 2xy - 3x^2 = 5$. As P passes through $(-1, 2)$, what are the rates of change with respect to x of y, of OP, and of the slope at P?

44. Time rates. If two variables x, y are connected by an equation of the form $y = F(x)$, then, § 40, their time rates of change dx/dt, dy/dt are connected by the equation $dy/dt = F'(x)dx/dt$. Also

If x, y are connected by an algebraic equation $f(x, y) = 0$, then x, y, dx/dt, dy/dt are connected by the equation got by differentiating $f(x, y) = 0$ with respect to t, regarding x, y as functions of t.

For since y is a function of x, and x of t, we have, §§ 40, 42,

$$\frac{d}{dt}f(x, y) = \frac{d}{dx}f(x, y)\frac{dx}{dt}$$
$$= \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right)\frac{dx}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

But $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx} = 0.$ Hence $\dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} = 0.$

In applying this theorem to problems, care must be taken not to substitute given values of x or y in $f(x, y) = 0$ before differentiating.

EXAMPLE 1. Water is running into a conical receiver 8 ft. deep and 8 ft. across at the top at the rate 4 cu. ft./min. How fast is the surface rising when the depth is 5 ft.?

The volume of the water is $V = \frac{1}{3}\pi r^2 h$

But $r = h/2$. Hence in terms of h $V = \frac{1}{12}\pi h^3$

Differentiating with respect to t $\dfrac{dV}{dt} = \dfrac{\pi}{4} h^2 \dfrac{dh}{dt}.$

Subst'g $\dfrac{dV}{dt} = 4,\ h = 5,$

we find $\dfrac{dh}{dt} = .204$ ft./min.

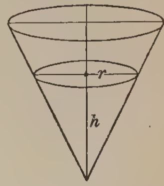

FIG. 13.

EXAMPLE 2. The ends of PQ, 13 in. long, move P on Ox, Q on Oy. When $OP = 5$ in., OP is increasing at the rate 3 in./sec.; at what rates are OQ and OPQ then changing?

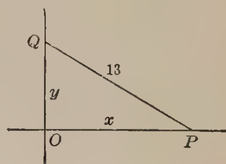

FIG. 14.

Let $x = OP,\ y = OQ,\ u = OPQ$. Then

(a) $x^2 + y^2 = 169$ (1)

$\therefore\ x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = 0$ (2)

When $x = 5$, (1) gives $y = 12$. Also $\dfrac{dx}{dt} = 3$ $\therefore$, by (2), $\dfrac{dy}{dt} = -\dfrac{5}{4}$

(b) $u = \dfrac{1}{2}xy$ $\therefore\ \dfrac{du}{dt} = \dfrac{1}{2}\left[y\dfrac{dx}{dt} + x\dfrac{dy}{dt}\right] = 14\frac{7}{8}.$

EXERCISE V

1. A man is approaching a flag pole 100 ft. high at the rate 3 ft./sec. At what rate is his distance from the top of the pole changing when he is 50 ft. from its foot?

2. P and Q move on Ox and Oy in such a manner that OPQ is always 10 sq. in. Find OP, OQ at the time when OP increases twice as fast as OQ decreases.

3. Sand poured on the ground is forming a conical pile whose altitude is always 2/3 the radius of its base. How fast is the sand falling if when the height of the pile is 3 ft. it is increasing at the rate 2 in./min.?

4. A baseball diamond is a square 90 ft. to a side. A batter is running to first base at the rate 25 ft./sec. When he has run halfway, at what rate is his distance from second base changing?

5. Two ships A and B are steaming eastward, at the rates 14 and 16 m./h. respectively. A starts at 1 P.M. and B at 3 P.M., B's starting point being 5 m. due south of A's. Are they approaching or separating at 6 P.M., and how fast?

6. A light A is on the ground 40 ft. from the wall BC of a building. A man 6 ft. tall is walking toward BC at the rate 4 ft./sec. At what rate is his shadow moving down BC when he is 35 feet from A?

7. A light A is at the top of a pole 60 ft. high. From a point on the ground 25 ft. from the base of the pole a ball B is tossed upward with an initial velocity of 30 ft./sec. At what rate is B's shadow moving along the ground one second later?

8. A point P is moving on the line $3x - 4y - 12 = 0$ with the constant speed 5 in./sec. How fast is OP changing when the x of P is -2?

9. An aeroplane which is flying due west at an altitude of 1 m. and with the velocity 100 m./h. passes vertically over an automobile which is running due north 40 m./h. At what rate are the two separating 1 hour later?

10. Gas is escaping from a spherical balloon at the rate 1 cu. ft./min. How fast is the radius r of the balloon decreasing, and how fast is its surface shrinking, when $r = 10$ ft.?

11. A ship is moving 600 ft./min. A man in its stern, and 20 ft. above the water, is pulling in, at the rate 200 ft./min., a rope attached to a row boat. When there is 52 ft. of rope out, at what rate is the boat running through the water?

12. A solution is filtering through a conical filter, 18 in. deep and 12 in. across at the top, into a cylindrical vessel whose diameter is 10 in. When the depth of the solution in the filter is 12 in., its level is falling at the rate 1 in./min. At what rate is its level in the cylinder then rising?

13. A light A is 20 ft. from a wall and 10 ft. above the center of a path which is at right angles to the wall. A man 6 ft. tall is walking on the path toward the wall at the rate 2 ft./sec. When he is 4 ft. from the wall, how fast is his shadow moving up it?

14. Two ships, A and B, leave the same port on the same day, A at noon and B at 2 P.M. If A runs 10 mi./h. on a course due north, and B runs 12 mi./h. on a course due east, how fast are they moving apart at 10 P.M.?

15. A point P is moving on the upper half of the circle $x^2 + y^2 - 4x = 0$. When the abscissa is 3 in., it is increasing at the rate 5 in./sec. At what rates are its ordinate and its distance from the origin then changing?

IV. MAXIMA AND MINIMA

45. Maximum and minimum values. Let $y = f(x)$ be a function which has a derivative, finite or infinite, for all values of x under consideration. If when x, increasing, passes through the value x_1, y ceases to increase and begins to decrease, then $f(x_1)$ is called a *maximum value* of y (as at A and C); if y ceases to decrease and begins to increase, then $f(x_1)$ is called a *minimum value* of y (as at B and D).[1]

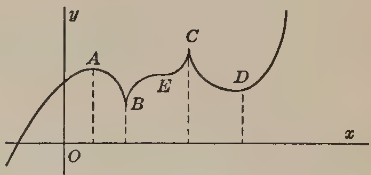

FIG. 15.

In both cases $f'(x_1)$ *is* 0 *or* ∞; for if $f'(x_1)$ were $+$, or $-$, y would continue to increase, or decrease, as x passed through the value x_1.

If as x passes through x_1, $f'(x)$ changes sign from $+$ to $-$, then $f(x_1)$ is a maximum; from $-$ to $+$, a minimum. Hence the following rule:

To find the maximum and minimum values of $y = f(x)$, determine all the real values x_1 of x for which $f'(x)$ is 0 *or* ∞. *Then for each such x_1 find whether or not $f'(x)$ changes sign as x increases through x_1.*

If sgn $f'(x)$ changes $+$ to $-$, $f(x_1)$ is a maximum value.

If sgn $f'(x)$ changes $-$ to $+$, $f(x_1)$ is a minimum value.

If sgn $f'(x)$ does not change, $f(x_1)$ is neither a maximum nor minimum.

[1] Hence a maximum value $f(x_1)$ may not be the greatest of all values of $f(x)$, but it is the greatest value in some interval extending to both sides of x_1; similarly for a minimum value.

The points of the graph of $y = f(x)$ at which y is a maximum or minimum and $f'(x)$ is 0 are called the *turning points* of the curve. Thus A and D in Fig. 15.

EXAMPLE 1. Find the maximum and minimum values of the function $y = x^3 - 6x^2 + 9x - 1$. Also its graph.

Solving $\quad f'(x) = 3x^2 - 12x + 9 = 0$
we get $x = 1$ or 3.

Hence $\quad f'(x) = 3(x-1)(x-3)$

As x increases through 1, sgn $f'(x)$ changes from $(-)(-) = +$ to $(+)(-) = -$; hence $f(1) = 3$ is a maximum value.

At $x = 3$, sgn $f'(x)$ changes $-$ to $+$; hence $f(3) = -1$ is a minimum value. In the x intervals $(-\infty, 1)$, $(1, 3)$, $(3, \infty)$, sgn $f'(x)$ is $+, -, +$; hence the graph in Fig. 16.

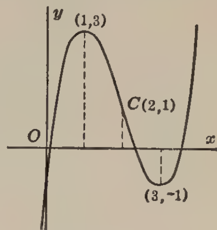

FIG. 16.

EXAMPLE 2. Investigate $y = (x-2)^{2/3} + 1$ for maximum and minimum values.

Here $f'(x) = 2/3(x-2)^{1/3}$, which at $x = 2$ becomes ∞ and changes sign $-$ to $+$; hence $f(2) = 1$ is a minimum value. In the x interval $(-\infty, 2)$, y decreases from ∞ to 1; in $(2, \infty)$, y increases from 1 to ∞.

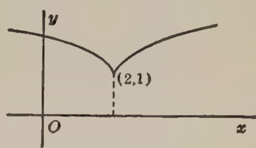

FIG. 17.

EXAMPLE 3. Find the maxima and minima of the following; also their graphs.

1. $y = x^2 - 4x + 3$
2. $y = 1 + 2x - x^2$
3. $y = x^5$
4. $y = x^3 + 2x^2 - 4x - 5$
5. $y = 2 + 3x + 4x^2 - x^3$
6. $y = \sqrt{2x - x^2}$

46. Concavity. Points of inflection. 1. Suppose that $f'(x)$ has a derivative $f''(x)$, finite or infinite, at every point P of the curve $y = f(x)$; and suppose the x of P to be increasing. By § 32,

When $f''(x)$ is $+$, the slope $f'(x)$ is increasing, hence the tangent at P is turning counter-clockwise and the part of the curve near P is *concave upward*. This is the case at any point but C on ABC in Fig. 18.

When $f''(x)$ is $-$, $f'(x)$ is decreasing, hence the tangent at P is turning clockwise and the part of the curve near P is *concave downward.* Thus at any point but C on CDE.

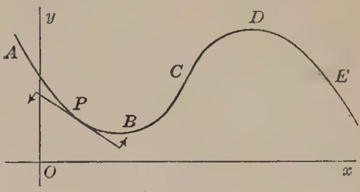

FIG. 18.

At a point where $f''(x)$ becomes 0 or ∞ and *changes sign*, the sense in which the tangent is turning is reversed. Thus at C. To one side of C the curve is concave upward, to the other concave downward, so that the curve crosses the tangent. Such a point C is called a *point of inflection.*

2. Evidently a point where $f'(x)$ is 0 and the curve is concave upward is a minimum point; concave downward, a maximum point. Hence

If $f'(x_1)$ is 0 and $f''(x_1)$ is $+$, then $f(x_1)$ is a minimum.
If $f'(x_1)$ is 0 and $f''(x_1)$ is $-$, then $f(x_1)$ is a maximum.

This test is less general but sometimes easier to apply than that of § 45.

EXAMPLE 1. In Ex. 1, § 45, we have $f''(x) = 6(x - 2)$ which is 0, $-$ to $+$, at $x = 2$. Hence the point of inflection $C(2, 1)$. At the maximum point B (1, 3) we have $f''(1) = -6$ *i.e.* negative; at the minimum D (3, 1), $f''(3) = 6$ *i.e.* positive.

EXAMPLE 2. Find the points of inflection of $y = x^3(x - 2)$.

$f'(x) = 2x^2(2x - 3)$ $\quad$ $f''(x) = 12x(x - 1)$

1. $f'(x)$ is 0 at $x = 0$ without changing sign, and at $x = 3/2$, $-$ to $+$.

Hence the only turning point is the minimum point C $(\frac{3}{2}, -\frac{27}{16})$.

2. $f''(x)$ is 0 at $x = 0$, $+$ to $-$, and at $x = 1$, $-$ to $+$. Hence the points of inflection $O(0, 0)$ and B $(1, -1)$, where the slopes, found from $f'(x)$, are 0 and -2. Sgn $f''(x)$ shows that the curve is concave upward from $x = -\infty$ to O, then downward to B, then upward.

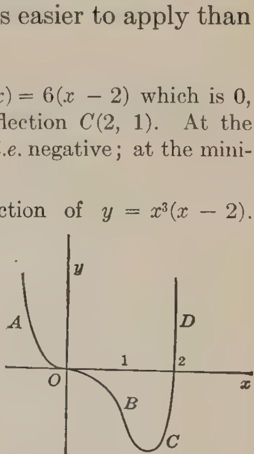

FIG. 19.

EXAMPLE 3. Find the turning points and points of inflection of the following; also their graphs.

1. $y = 3x - x^3$
2. $y = x^4$
3. $y = x^{1/3}$
4. $y = (x^2 - 1)^2$
5. $y = 6x^2 - 4 - x^4$
6. $y = x^3 + 6x$
7. $2y = 2x^3 - 3x^2 - 12x + 6$
8. $3y = x^3 - 3x^2 - 9x + 15$
9. $y = x^4 - 2x^3 + 2x - 1$

47. Greatest and least values. 1. If $f(x)$ is continuous in the interval (ab), it has a greatest value M in (ab), § 18. It may take the value M either at $x = a$ or b or at some point $x = x_1$ between a and b. In the latter case, $M = f(x_1)$ is the greatest of the maximum values of $f(x)$ in (ab). Similarly for the least value m of $f(x)$ in (ab).

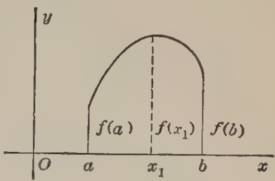

FIG. 20.

2. *If $f'(x)$ exists throughout (ab) and is 0 or ∞ at one and but one point $x = x_1$ between a and b, then*

$$f(x_1) = M, \text{ if } f(x_1) > f(a), f(b); \quad f(x_1) = m, \text{ if } f(x_1) < f(a), f(b)$$

48. Problems. Each of the following problems is concerned with some magnitude which varies continuously subject to certain restrictions, and its greatest or least value is sought. The given conditions enable one to express the numerical measure of the magnitude as a function of a single variable in some interval (ab). We seek the greatest or least value of the function in this interval.

EXAMPLE 1. Find the largest box open at the top that can be made of a piece of tin a in. square by cutting out equal square pieces at the corners and bending upward the projecting portions which remain.

Let x be the height of the box; it is restricted to values between 0 and $a/2$.

Then if u denote the volume of the box, $\quad u = x(a - 2x)^2$

Hence, diff'g and simplifying, $\quad \dfrac{du}{dx} = (a - 6x)(a - 2x)$

At $x = a/6$, du/dx vanishes, $+$ to $-$; and $a/6$ is between 0 and $a/2$. Hence $a/6$ is the height of the largest box. Its volume is $2\,a^3/27$. This also follows from § 47, 2.

At $x = a/2$, du/dx also vanishes, $-$ to $+$, but as this value of x is not between 0 and $a/2$ it is to be discarded. It gives, of course, a minimum value of the unrestricted function $x(a - 2\,x)^2$, namely the value 0. The box is nonexistent for this value.

Example 2. What points of the parabola $y^2 = 4\,ax$ are nearest the point $C(c, 0)$?

$$CP^2 = (x - c)^2 + y^2 = (x - c)^2 + 4\,ax \qquad (x > 0)$$

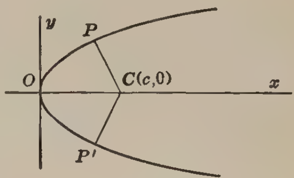

Fig. 21.

Since CP is least when CP^2 is least, we seek the least value of

$$u = (x - c)^2 + 4\,ax \qquad (x > 0)$$

$$\frac{du}{dx} = 2(x - c) + 4\,a \text{ vanishes, } - \text{ to } +,$$

at $x = c - 2\,a$

Hence $x = c - 2\,a$ is the abscissa of the two curve points nearest C when $x = c - 2\,a > 0$, *i.e.* when $c > 2\,a$. But when $c \leqq 2\,a$, then $du/dx = 2[x + (2\,a - c)]$ is always $+$ for $x > 0$ $\quad\therefore$ CP takes its least value when $x = 0$, and O is the curve point nearest C.

Example 3. Find 1. the cylinder of greatest volume inscribed in a given right circular cone; 2. the cylinder of greatest curved surface; 3. the cylinder of greatest total surface.

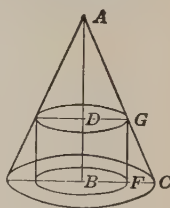

Fig. 22.

1. Let $r = BC$, $a = BA$, $x = BF$, $y = BD$ $\therefore$ $V = \pi x^2 y$. We must express V in terms of x (or y) only. But

$$\frac{FG}{FC} = \frac{BA}{BC} \quad \therefore \frac{y}{r - x} = \frac{a}{r} \qquad \therefore y = \frac{a}{r}(r - x).$$

Hence $V = \dfrac{\pi a}{r} x^2(r - x)$ $\quad (0 < x < r)$, which will be found to take its greatest value when $x = (2/3)r$.

2. The area of the curved surface $S_1 = 2\,\pi xy = (2\,\pi a/r)(rx - x^2)$ is greatest when $x = r/2$.

3. The total area $S_2 = 2\,\pi[(a/r)(rx - x^2) + x^2]$ is greatest when $x = ar/2(a - r)$. But we must have $ar/2(a - r) < r$ $\quad\therefore$ $r < a/2$. If $r \geqq a/2$, then as x increases from 0 to r, S_2 increases from 0 to $2\,\pi r^2$.

EXERCISE VI

1. From a point on the hypotenuse of a right-angled triangle perpendiculars are drawn to the other two sides. When is the rectangle thus formed a maximum?

2. Prove that a square is the greatest rectangle of given perimeter; also that it is the greatest rectangle that can be inscribed in a circle.

3. The total area of a page is 96 sq. in., the combined widths of the margins at the top and bottom 3 in., at the sides 2 in. For what dimensions of the page is the printed area greatest?

4. What is the least number of sq. ft. of lumber that will form an open tank with a square base and a capacity of 108 cu. ft.?

5. For what height will the cost of a box with square base and 324 cu. ft. capacity be least, if the bottom costs 4 cts., the top 5 cts., and the sides 3 cts. per sq. in.?

6. The axes Ox, Oy are met at P, Q by a variable line through the point (2, 3). Find the triangle OPQ of least area.

7. The sides AB, AC of a triangle ABC are met by a parallel to BC in the points D, E, and these points are joined to a fixed point F in BC. For what position of the line DE will the triangle DEF be a maximum?

8. Find the greatest rectangle that can be inscribed in the ellipse $x^2 + 4y^2 = 4$.

9. Find greatest rectangle that can be inscribed in the space bounded by $y^2 = 8x$ and $x = 4$.

10. A ship A is sailing from E. to W., 8 m./h., and a ship B from N. to S., 7 m./h. At noon A is 10 m. east of B's course and B is 14 m. north of A's course. When will their distance apart be least?

11. A manufacturer can now ship a cargo of 100 tons at a profit of $5 per ton. He estimates, however, that by waiting he can add 20 tons per week to the shipment, but that the profit on all that he ships will be reduced 25 cts. per ton per week. How long will it be to his advantage to wait?

12. A man is in a rowboat 3 m. from the nearest point of the shore and wishes to reach a point of the shore 6 m. from this nearest point as quickly as possible. He thinks that he can row 4 m./h. and move on shore 5 m./h. To what point of the shore should he row?

13. Find (1) the cylinder of greatest volume that can be inscribed in a sphere of radius a; (2) the cylinder of greatest surface; (3) the cone of greatest volume.

14. The cone K has the altitude a, and r is the radius of its base B. Find the greatest cone K' which has the center of B for vertex and the circle in which K is cut by some plane parallel to B for its base.

15. The points A and B are to the same side of a given line. For what point P on the line is $AP^2 + BP^2$ least? For what point is $AP + BP$ least?

16. A line drawn from the corner A of the rectangle $ABCD$ meets the side BC at E and DC produced at F. When is the sum of the triangles ABE, FCE least?

17. A point A is at the distance h above the center C of a sphere of radius r. A cone is constructed having A for vertex, and for base the circle got by cutting the sphere by a plane perpendicular to CA. When is this cone largest?[1]

18. A sheet of paper $ABCD$ has the width $AB = a$. A triangle BEF is formed by folding the sheet in such a manner that the corner B falls on the edge AD. When is the area of BEF least? When is the length of EF least?

19. What points of the ellipse $x^2 + 4y^2 = 8$ are nearest the point $(1, 0)$?

20. How high above the center C of a circular plot of grass 30 ft. across should an electric light A be placed to produce the greatest brightness at points P on the circumference, it being assumed that the brightness at P varies inversely as AP^2 and directly as $\sin CPA$?

21. If the cost of propelling a boat varies as the cube of the speed generated, what is the most economical speed against a 3 m./h. current?

22. Let A, B, C be three fixed points on a given straight line; find the point P the sum of the squares of whose distances from A, B, C is least.

23. If the perimeter of an isosceles triangle be given, show that the area is greatest when the triangle is equilateral.

24. Let $A(a, 0)$ and $B(-a, 0)$ be two fixed points on the x-axis, and P, Q two movable points, P on the lower half of the y-axis, and Q vertically below A. If $BP + PA + AQ$ is constant, for what position of P is the midpoint of PQ as far as possible from the x-axis?

25. Let A be a fixed point on the x-axis, and P, Q variable points, P on the x- and Q on the y-axis, and such that $AP = PQ$. Prove that the area of the triangle OPQ is greatest when OP is one third of OA.

[1] The plane may be above C or below C.

26. If the cost of fuel for running a steamboat be proportional to the square of the speed generated and is \$8 an hour when a speed of 8 m./h. is generated, what is the most economical speed against a 5 m./h. current, the fixed charges being \$18 per hour?

49. Implicit functions. The maximum and minimum values of a function which is given by two equations of the form $u = \phi(x, y), f(x, y) = 0$ may be found as follows:

EXAMPLE 1. Find the greatest rectangle that can be inscribed in a circle of radius a.

We have $u = 4\,xy$ (1) where $x^2 + y^2 = a^2$ (2)

Eq's (1), (2) define u as a function of x: $u = 4\,x\sqrt{a^2 - x^2}$ (3)

But we may solve the problem without using (3); for differentiating (1) and (2), regarding y and u as functions of x, and setting $\frac{du}{dx} = 0$,

$$y + x\frac{dy}{dx} = 0 \qquad x + y\frac{dy}{dx} = 0$$

Hence, eliminating $\frac{dy}{dx}$, $y^2 - x^2 = 0$

$$\therefore y = x \quad (4)$$

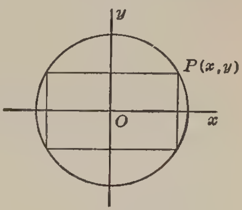

FIG. 23.

Evidently the rectangle thus obtained is a square with half side $x = a/\sqrt{2}$, and area $u = 2\,a^2$. That it is the greatest rectangle follows from § 47, 2. For $x = a/\sqrt{2}$ is between the extreme values, 0 and a, of x, and is the only value of x between 0 and a for which du/dx is 0. And the value of u for $x = a/\sqrt{2}$, namely $2\,a^2$, is greater than those for $x = 0$ and $x = a$.

EXAMPLE 2. Find when the surface of a cylindrical tank of given content will be least (1) if the tank be completely enclosed; (2) if it be open at the top.

EXAMPLE 3. For what dimensions will the least material be required to construct a conical tent of given content?

EXAMPLE 4. Through the point (a, b) a line is drawn which meets Ox at P and Oy at Q. What is the least length that PQ can have?

EXAMPLE 5. Find the dimensions of the cone of greatest curved surface that can be inscribed in a given sphere.

EXAMPLE 6. Show that the maximum and minimum values of y, when defined as a function of x by an equation of the form $f(x, y) = 0$, may be found by solving the equations $f(x, y) = 0$, $\partial f/\partial x = 0$.

EXAMPLE 7. If $x^4 - x^2y + y^3 = 0$, find the minimum value of y.

50. Roots of algebraic equations. 1. The real roots c of an algebraic equation $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ are pictured by the points where the graph of $y = f(x)$ meets Ox. When we can solve the equation $f'(x) = 0$, we can readily make a rough determination of this graph as in §§ 45, 46. The curve thus got will at once indicate the whereabouts of the roots c; it will also suggest what values of $f(x)$ to compute in order to locate each root more precisely — by aid of the theorem (§ 18, 2):

If $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has at least one root between a and b.

Example 1. Locate the roots of $f(x) = x^3 - 3x + 1 = 0$, using the graph of $y = f(x)$. $f'(x) = 3x^2 - 3 = 0$ gives the turning points $C(-1, 3)$, $F(1, -1)$. Since C and F are on opposite sides of Ox, there is a root E between -1 and 1; and since $f(0) = 1$ is $+$, E is between 0 and 1. Since $f(-1)$ is $+$ and $f(-\infty)$ is $-$, there is a root B to the left of -1; and since $f(-2) = -1$ is $-$, B is between -2 and -1. Similarly, since $f(1)$ is $-$ and $f(2) = 3$ is $+$, the third root G is between 1 and 2.

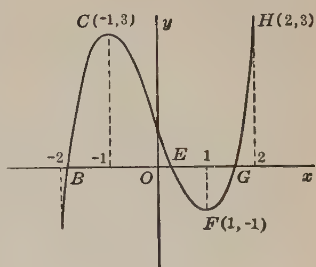

Fig. 24.

Moreover the line FH will cut the line segment 1, 2 in the ratio $F1/2H = 1/3$; hence G is between $1\frac{1}{4}$ and 2.

2. If $f(c)$ is 0, then $x - c$ is a factor of $f(x)$; and conversely. If $f(x)$ contains $(x - c)^r$, $r \geqq 1$, and no higher power of $x - c$, as a factor, then c is called a *root of order r*. We also call c a *simple root* when $r = 1$, a *multiple root* when $r > 1$. It is easy to show [1] that

If c is a simple root of $f(x) = 0$, it is not a root of $f'(x) = 0$; but if c is a root of order $r > 1$ of $f(x) = 0$, it is a root of order $r - 1$ of $f'(x) = 0$.

[1] For if $f(x) = (x - c)^r\phi(x)$, then $f'(x) = r(x - c)^{r-1}\phi(x) + (x - c)^r\phi'(x) = (x - c)^{r-1}[r\phi(x) + (x - c)\phi'(x)]$.

Hence the graph of $y = f(x)$ merely cuts Ox at $x = c$ when $r = 1$, but touches Ox when $r > 1$, also crossing Ox when r is odd since in that case $(x - c)^r$ changes sign at $x = c$.

EXAMPLE 2. $f(x) = (x + 1)(x - 1)^2(x - 2)^3 = 0$ has the roots -1, 1, 2; their orders are 1, 2, 3. $f'(x) = (x - 1)(x - 2)^2(6x^2 - 5x - 5)$ does not contain $x + 1$ but does contain $x - 1$ and $(x - 2)^2$. Draw a rough graph of $y = f(x)$ showing that it crosses Ox at $x = -1$, touches at $x = 1$, touches and crosses at $x = 2$.

EXAMPLE 3. Show that if the degree n of $f(x) = a_0x^n + \cdots + a_n$ is odd, then $f(-\infty)$ and $f(\infty)$ have opposite signs and therefore that the sum of the orders of the real roots c of $f(x) = 0$ is odd, being at least one. How is it when n is even?

EXAMPLE 4. Draw a typical graph of $y = f(x)$ for the case $n = 5$ and a_0 positive, showing that the number of times it crosses Ox may be 5, 3, or 1.

EXAMPLE 5. Show that between two points where the curve $y = f(x)$ meets Ox there is at least one turning point, and between two turning points at least one point of inflection.

EXAMPLE 6. Show that $x^3 - 3px + q = 0$ has three real roots when $q^2 < 4p^3$.

EXAMPLE 7. Draw graphs of the following, locating the points where each curve meets Ox.

1. $y = x^2(x - 2)^3$
2. $2y = 2x^3 - 3x^2 - 12x + 6$
3. $y = x^4 + 4x^3 - 8x^2 + 2$

51. Graphs of rational fractional functions of x. Every such function can be reduced to the form:

$$F(x) = \frac{f(x)}{\phi(x)} = \frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^p + b_1x^{p-1} + \cdots + b_p} \tag{1}$$

where $f(x)$ and $\phi(x)$ have no common factor.

Evidently $F(x)$ is 0 when $f(x)$ is 0; $F(x)$ is ∞ when $\phi(x)$ is 0; and for all other finite values of x, $f(x)$ is continuous and not 0. Let $|x| \to \infty$. Then, § 11,

$$F(x) \to 0 \text{ when } n < p \qquad F(x) \to \frac{a_0}{b_0} \text{ when } n = p$$

$$|F(x)| \to \infty \text{ when } n > p$$

As the cases $y = 1/x$, $y = 1/x^2$, p. 10, Ex. 1, illustrate, the graphs of fractional functions $y = F(x)$ may have "asymptotes."

If as a point P moves out on an infinite curve branch C it tends to move along a definite line l, so that the distance $DP \rightarrow 0$ and the slope at $P \rightarrow$ the slope of l, then l is called an *asymptote* of C.

1. If $x - c$ is a factor of $\phi(x)$ in (1), c being real, then the line $x - c = 0$ is an asymptote (like l') of the curve $y = F(x)$.

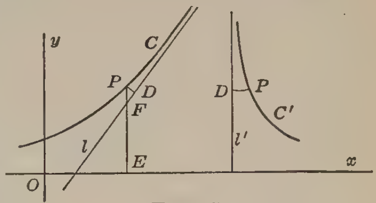

Fig. 25.

2. If $n - p \gtreqless 1$ in (1), we can reduce $y = F(x)$ to the form $y = mx + b + \psi(x)$, where m, b, one or both, may be 0, and $\psi(x)$ denotes a *proper*[1] fraction, so that when $|x| \rightarrow \infty$ then $\psi(x) \rightarrow 0$ and also, as it is easy to show, $\psi'(x) \rightarrow 0$. Hence, by the following theorem, $y = mx + b$ is an asymptote (like l) of $y = F(x)$.

Thus $y = \dfrac{x^2}{x - 1} = x + 1 + \dfrac{1}{x - 1}$; hence $y = x + 1$ is an asymptote. So also is $x - 1 = 0$, by 1.

If $F(x) = mx + b + \psi(x)$, and $\psi(x)$ and $\psi'(x) \rightarrow 0$ when $|x| \rightarrow \infty$, then the line $y = mx + b$ is an asymptote of the curve $y = F(x)$.

For let l in Fig. 25 be the line $y = mx + b$ and C part of the curve $y = F(x)$. Then for $x = OE$, we have $mx + b + \psi(x) = EP$, and $mx + b = EF$ $\therefore$ $FP = \psi(x)$.

Hence when $\psi(x) \rightarrow 0$, then $FP \rightarrow 0$ and therefore also $DP = FP \sin DFP \rightarrow 0$.

Also $F'(x) = m + \psi'(x) \rightarrow m$, since $\psi'(x) \rightarrow 0$.

[1] A rational fraction in x is called a *proper fraction* when its numerator is of lower degree than its denominator.

Observe that for sufficiently great values of $|x|$, C *is above or below* l *according as* $\psi(x) = FP$ *is positive or negative.*[1]

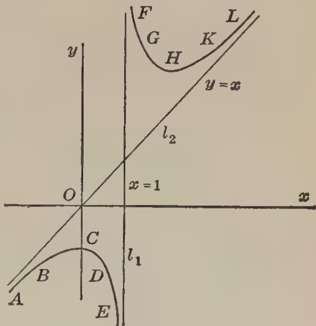

FIG. 26.

EXAMPLE 1. Find the graph of $y = (x^2 - x + 1)/(x - 1)$.

1. Since y becomes infinite at $x = 1$, $-$ to $+$, the line $x - 1 = 0$ (l_1) is an asymptote to a branch DE below (at left) and to a branch FG above (at right).

2. By division, $y = x + \dfrac{1}{x-1}$ and $1/(x-1) \to 0$ when $|x| \to \infty$.

Hence the line $y = x(l_2)$ is an asymptote at $x = -\infty$ to a branch AB (below l_2 since $1/(x-1)$ is $-$ as $x \to -\infty$) and at $x = \infty$ to a branch KL above l_2.

3. We find $F'(x) = x(x-2)/(x-1)^2$ which vanishes at $x = 0$, $+$ to $-$, and at $x = 2$, $-$ to $+$, and gives the turning points $C(0, -1)$ and $H(2, 3)$. The asymptotes and the sign of $F'(x)$ show that as x increases from $-\infty$ to ∞ the curve follows the course $ABCDEFGHKL$.

EXAMPLE 2. Find the graph of $y = x/(x^2 - 5x + 4)$.

$$F(x) = \frac{x}{(x-1)(x-4)} \qquad \therefore\ F'(x) = -\frac{(x+2)(x-2)}{(x^2 - 5x + 4)^2}$$

The curve cuts Ox at $x = 0$. It has the vertical asymptotes $x - 1 = 0$ and $x - 4 = 0$. Since $F(x)$ is a proper fraction its asymptote l of Fig. 25 is $y = 0$. Sgn $F(x)$ is $-$ in the x interval $(-\infty, 0)$, $+$ in $(0, 1)$, $-$ in $(1, 4)$, $+$ in $(4, \infty)$. These facts suffice to show that the curve must be of the type indicated in Fig. 27. To determine it more precisely we find the turning points given by $F'(x) = 0$. They are $B(-2, -1/9)$ and $E(2, -1)$.

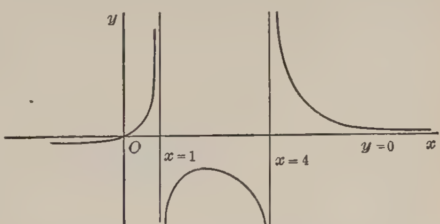

FIG. 27.

[1] Since $FP = y - mx - b$, we have the theorem: When the point $P(x, y)$ is not on the line $y - mx - b = 0$, then $y - mx - b$ represents the distance FP from the line to P measured parallel to Oy. It is $+$ or $-$ according as P is above or below the line.

EXERCISE VII

Find the graphs of the following (1–12):

1. $y = \dfrac{x-2}{x+1}$ 2. $y = \dfrac{2x^2 - 3x}{x-2}$ 3. $y = \dfrac{x^2-1}{x^2-4}$

4. $y = \dfrac{x^2+x-2}{x^2-x-2}$ 5. $y = \dfrac{2x^2}{x^2-1}$ 6. $y = \dfrac{2x}{x^2+1}$

7. $y = \dfrac{x^2+2x-3}{x^2-8x+12}$ 8. $y = \dfrac{x^3}{(x-1)^2}$ 9. $y = \dfrac{x^3}{x^2+1}$

10. $y = \dfrac{x^3}{x^2-1}$ 11. $y = \dfrac{2x^2-x}{x^2-1}$ 12. $y = \dfrac{x^3}{x^2-x-6}$

13. Prove that when $(x - \beta)^r$, $r > 1$, is a factor of $f(x)$, the curve $y = f(x)/\phi(x)$ touches the x-axis at $x = \beta$, crossing it or not according as r is odd or even.

14. Prove that when $(x - \beta)^r$, $r \gtreqless 1$, is a factor of $\phi(x)$, the infinite value taken by $y = f(x)/\phi(x)$ at $x = \beta$ is a maximum or minimum if r is even, but not if r is odd; also that the sense in which the tangent to the curve turns as x increases changes at $x = \beta$ if r is odd but not if r is even.

15. Show that two of the roots of $x^3 - 3px + q = 0$ are equal if $q^2 = 4p^3$.

V. INVERSE FUNCTIONS

52. Theorem. *If between $x = a$ and $x = b$, $y = f(x)$ is a one valued, continuous, and increasing function of x, then also between $x = a$ and $x = b$, x is a one valued, continuous, and increasing function of y.*

1. For since $f(x)$ is continuous in the x-interval (a, b), and $f(a)$ and $f(b)$ are its least and greatest values in (a, b), to each value y_1 of y between $f(a)$ and $f(b)$ corresponds a value x_1 of x such that $f(x_1) = y_1$, § 18, 2.; and there can be but one such value since $f(x)$ is an increasing function and therefore can take no value y_1 more than once. Hence, by the definition of function, § 14, between $x = a$ and $x = b$, x is a one valued function of y, namely a function $x = \phi(y)$ such that

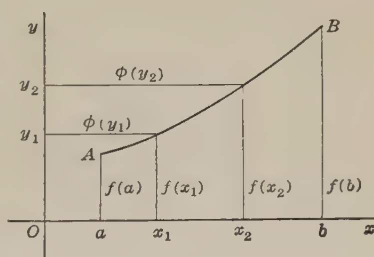

Fig. 28.

$$\text{If } y_1 = f(x_1) \text{ then } x_1 = \phi(y_1) \tag{1}$$

2. The function $x = \phi(y)$ *increases* with y. If $y_1 < y_2$, then $x_1 < x_2$; for were $x_1 \geqq x_2$, then $y_1 \geqq y_2$.

3. The function $x = \phi(y)$ is *continuous*. For let x_1, y_1 be corresponding values of x, y as in (1); it is to be proved that $\lim_{y \to y_1} \phi(y) = \phi(y_1)$. Compare § 8.

First let y increase toward y_1 as limit. Then $x = \phi(y)$ increases but remains $< \phi(y_1)$ and therefore (§ 5) approaches some limit $l(\leqq \phi(y_1))$. But, since $f(x)$ is continuous, when

$x \to l$, then $f(x) \to f(l)$. Hence $f(l) = y_1$ and therefore $l = \phi(y_1)$.

Similarly, when $y \to y_1$, then $\phi(y) \to \phi(y_1)$. Therefore

$$\lim_{y \to y_1} \phi(y) = \phi(y_1).$$

Similarly, if between $x = a$ and $x = b$, $y = f(x)$ is a one valued, continuous, and decreasing function of x, then also between $x = a$ and $x = b$, x is a one valued, continuous, and decreasing function of y.

53. The inverse of a function. The function $x = \phi(y)$ is called the *inverse* of the function $y = f(x)$ in the x-interval (a, b).

If instead of increasing or decreasing throughout (a, b), $y = f(x)$ first increases to a maximum value at $x = c$, say, and then decreases, $y = f(x)$ has one inverse $x = \phi_1(y)$ in the interval (a, c), and another, $x = \phi_2(y)$, in (c, b); and so in general. If the equation $y = f(x)$ can be solved for x in terms of y, these inverses are the several solutions.

EXAMPLE 1. The inverses of the function $y = (x^2 - 1)^2$, got by solving for x, are

1. $x = -(1 + y^{1/2})^{1/2}$, x in $(-\infty, -1)$
2. $x = -(1 - y^{1/2})^{1/2}$, x in $(-1, 0)$
3. $x = (1 - y^{1/2})^{1/2}$, x in $(0, 1)$
4. $x = (1 + y^{1/2})^{1/2}$, x in $(1, \infty)$

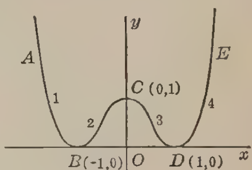

FIG. 29.

The graphs of these four inverses are the parts of the curve $y = (x^2 - 1)^2$ marked 1, 2, 3, 4. They are separated at the points B, C, D where $f'(x)$ vanishes and changes sign.

EXAMPLE 2. Find the inverses of the following and their graphs:

1. $y = 3x - 2$ **2.** $y = x^2$ **3.** $y = x^2 - 2x$ **4.** $y = x/(x + 2)$

EXAMPLE 3. Find the graphs of the inverses of $y = x^3 - 4x^2 + 4x$.

54. The derivatives dy/dx and dx/dy. If $y = f(x)$ has a derivative $f'(x)$ which is continuous and not 0 at $x = x_1$, then $y = f(x)$ is an increasing or a decreasing function in the

neighborhood of the point (x_1, y_1), § 31, and therefore has an inverse $x = \phi(y)$ in that neighborhood, §§ 52, 53.

The function $x = \phi(y)$ *has a derivative* $\phi'(y)$ *at* (x_1, y_1)*; and*

$$\phi'(y_1) = 1/f'(x_1) \tag{1}$$

For $\Delta x/\Delta y = 1 \div \Delta y/\Delta x$, and $\Delta x \to 0$ when $\Delta y \to 0$, § 52, 3.; hence

$$\lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = 1 \Big/ \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta x} = 1 \Big/ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}, \text{ or } \phi'(y_1) = 1/f'(x_1)$$

We have $f'(x_1) = dy_1/dx_1$, $\phi'(y_1) = dx_1/dy_1$; hence, omitting subscripts,

$$\frac{dx}{dy} = 1 \Big/ \frac{dy}{dx}, \textit{ when } \frac{dy}{dx} \textit{ is continuous and not } 0 \tag{2}$$

Example. Thus $y = x^2$ gives $dy/dx = 2x$; the inverse $x = y^{1/2}$ gives $dx/dy = 1/2\, y^{1/2} = 1/2\, x$.

55. The derivative dy/dx when $x = \phi(t), y = \psi(t)$. Suppose that x and y are given as functions of another variable t by equations of the form $x = \phi(t)$, $y = \psi(t)$, where $\phi(t)$ and $\psi(t)$ are one valued and have continuous derivatives for all values of t under consideration. In the neighborhood of any value of t for which $\phi'(t) \neq 0$, the function $x = \phi(t)$ has an inverse $t = f(x)$, § 54; and the equations $y = \psi(t)$, $t = f(x)$ combined define y as a function of x. Hence, §§ 40, 54,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{\psi'(t)}{\phi'(t)}, \textit{ when } \phi'(t) \neq 0 \tag{3}$$

56. Parametric equations of curves. To any assigned value of t correspond definite values of $x = \phi(t)$ and $y = \psi(t)$ and therefore a definite point P of which these values are the coordinates. When t varies continuously, P will trace a curve. The equations $x = \phi(t)$, $y = \psi(t)$ are called the *parametric equations* of this curve in terms of the *parameter* t. Its slope at the point (x_1, y_1) corresponding to $t = t_1$ can be found by the formula § 55 (3).

EXAMPLE 1. The equations $x = t^2$, $y = 2t$ represent the parabola $y^2 = 4x$. For if we eliminate t between $x = t^2$, $y = 2t$, we get $y^2 = 4x$. As t increases, $-\infty$ to ∞, the entire curve is traced in the sense AOB. At the point $t = 1$, we have $x = 1$, $y = 2$, $dx/dt = 2t = 2$, $dy/dt = 2$. Hence $dy/dx = 2/2 = 1$, and the tangent at the point is $y-2=(x-1)$.

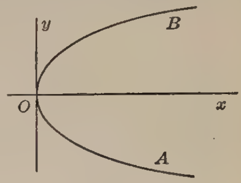

FIG. 30.

EXAMPLE 2. Find the tangent to $x = t^2$, $y = t^3$ at $t = -2$. Find the x, y equation of the curve. Trace it.

EXAMPLE 3. Show that $x = at + b$, $y = ct + d$ represent a straight line through the point (b, d) and having the slope c/a.

EXERCISE VIII

1. If $y = x^3 - 2x$, find dx/dy when $x = 2$. At what points is $dx/dy = 7$?

2. Find the inverses of $y = x^2 + 2x - 3$ and their graphs.

3. Find the tangents to the following curves at the points indicated.

   1. $x = 3t^2 + 2t$, $y = 1/t$, at $t = -2$
   2. $x = 25 - t^2$, $y = t^3$, at $t = 4$

4. Find the x, y equation of the curve $x = 1/t$, $y = 2 - t$. Trace the curve as t increases from $-\infty$ to ∞.

5. Show that $x = \dfrac{t^2 - 1}{t^2 + 1}$, $y = \dfrac{2t}{t^2 + 1}$ represent the circle $x^2 + y^2 = 1$.

6. Verify the following: $\dfrac{d^2x}{dy^2} = \dfrac{d}{dx}\left[1\Big/\dfrac{dy}{dx}\right]\dfrac{dx}{dy} = -\dfrac{d^2y}{dx^2}\Big/\left(\dfrac{dy}{dx}\right)^3.$

7. Show, if $x = \phi(t)$, $y = \psi(t)$, that
$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dt}\Big/\frac{dx}{dt}\right]\frac{dt}{dx} = \left[\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right]\Big/\left(\frac{dx}{dt}\right)^3$$

8. By aid of 6. and 7. find the points of inflection of $x = y^3 - 3y^2$, and $x = t^2 - t$, $y = t^3 + t$.

9. A steamship is moving through the water 30 ft./sec. A ball is dropped from a point of a mast 64 ft. vertically above a point A on the deck. Neglecting the resistance of the air, show that the path of the ball is an arc of a parabola. At what angle will it strike the deck?

VI. THE TRIGONOMETRIC FUNCTIONS

57. Length of a curve arc. A curve arc is called *convex* when the line joining any two of its points lies wholly within the space bounded by the arc and its chord.

In the convex arc AB inscribe a variable polygon $AP_1P_2 \cdots P_nB$, the lengths of whose sides all approach the limit 0 as n increases. It will be proved later that as $n \to \infty$ the length of the polygon $AP_1P_2 \cdots P_nB$ approaches a definite limit. This limit is called the *length of the arc* AB.

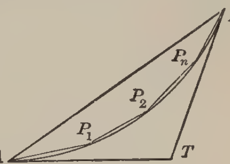

FIG. 31.

Join AB, and draw the tangents AT, BT. By elementary geometry, and § 5,

$$AB < AP_1P_2 \cdots P_nB < ATB; \quad \text{hence} \quad AB < \text{arc } AB < ATB$$

Suppose that B is made to move along the curve toward A as limit. The ratio ATB/AB will approach 1 as limit.[1] Hence the ratio arc AB/AB, which lies between ATB/AB and 1, will also approach 1 as limit. Therefore

The limit of the ratio of a convex arc to its chord, when the arc approaches 0, *is* 1.

58. The circle. Let the arc AB be the circumference of a circle of radius r. The perimeter of an inscribed regular

[1] For $AT + TB = AD \sec\theta + DB \sec\phi = AB \sec\theta + DB(\sec\phi - \sec\theta)$

$$\therefore \qquad \frac{ATB}{AB} = \sec\theta + \frac{DB}{AB}(\sec\phi - \sec\theta)$$

Let $B \to A$. Then $\theta, \phi \to 0$ $\quad \therefore \sec\theta, \sec\phi \to 1$

$$\therefore \frac{ATB}{AB} \to 1.$$

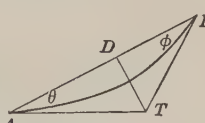

FIG. 32.

polygon of 4, 8, 16, $\cdots$, 2^n sides can be computed in terms of r. The limit which it approaches when $n \to \infty$ is $2\pi r$, where π denotes an irrational number whose value to the fifth decimal figure is 3.14159. Hence the length of the circumference of a circle of radius r is

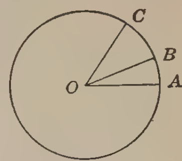

FIG. 33.

$$c = 2\pi r \qquad \pi = 3.14159 \cdots$$

59. Circular measure of angle. Let AOB be any angle. With O as center and any radius r describe a circle cutting OA and OB at A and B. Let l denote the length of the arc AB. It follows from the definition of length of circular arc that the ratio l/r is the same for all values of r. This ratio is called the *circular measure* of the angle AOB. Let θ denote its value. Then

$$\theta = l/r \qquad \text{and} \qquad l = r\theta \tag{1}$$

When $l = r$, we have $\theta = 1$. Hence the *unit of circular measure* is the angle AOC subtended by an arc AC equal to r. This angle is called the *radian*. And since $AOB : AOC = \text{arc } AB : \text{arc } AC = l : r$, the circular measure θ of AOB is the ratio of AOB to the radian AOC.

When $AOB = 360°$, we have $l = 2\pi r$ $\therefore \theta = 2\pi$. Hence the circular measures of 360°, 180°, 90°, 1° are 2π, π, $\pi/2$, $\pi/180$. Conversely a radian $= (180/\pi)°$.

In the calculus the measure of an angle always means its circular measure.

60. Theorem. *When $\theta \to 0$, the ratio $\sin\theta/\theta \to 1$.*

For $$\frac{\sin\theta}{\theta} = \frac{2r\sin\theta}{2r\theta} = \frac{\text{chd } B'B}{\text{arc } B'B}$$

But $$\lim_{\theta\to 0}\frac{\text{chd } B'B}{\text{arc } B'B} = 1, \ \S\ 57.$$

Hence $$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

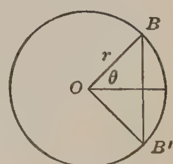

FIG. 34.

61. Continuity of the trigonometric functions. 1. The functions $\sin\theta$ and $\cos\theta$ are continuous for all finite values of θ. For in the identity

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \tag{1}$$

set $A = \theta + \Delta\theta$, $B = \theta$. We obtain

$$\sin(\theta + \Delta\theta) - \sin\theta = 2\cos\left(\theta + \frac{\Delta\theta}{2}\right)\sin\frac{\Delta\theta}{2} \tag{2}$$

Here $\cos\left(\theta + \frac{\Delta\theta}{2}\right)$ is between -1 and 1, and $\sin\frac{\Delta\theta}{2} \to 0$ when $\Delta\theta \to 0$, § 60. Hence

$$\lim_{\Delta\theta\to 0}[\sin(\theta + \Delta\theta) - \sin\theta] = 0 \tag{3}$$

The continuity of $\cos\theta$ follows from that of $\sin\theta$; for

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right).$$

2. It follows from 1. that $\tan\theta = \sin\theta/\cos\theta$ and $\sec\theta = 1/\cos\theta$ are continuous except when $\cos\theta = 0$, that is, when θ is some odd multiple of $\pi/2$; also that $\cot\theta = \cos\theta/\sin\theta$ and $\operatorname{cosec}\theta = 1/\sin\theta$ are continuous except when $\sin\theta = 0$, that is, when θ is 0 or some even multiple of $\pi/2$.

62. Derivative of sin *x*. 1. Let x denote the circular measure of any angle. If $y = \sin x$, then

$$\Delta y = \sin(x + \Delta x) - \sin x$$

$$= 2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2} \qquad [\S\ 61\ (2)]$$

and

$$\frac{\Delta y}{\Delta x} = \cos\left(x + \frac{\Delta x}{2}\right)\frac{\sin(\Delta x/2)}{\Delta x/2}$$

When $\Delta x \to 0$, the first factor on the right approaches the limit $\cos x$, § 61, and the second factor approaches the limit 1, (§ 60). Hence

$$\frac{d}{dx}\sin x = \cos x$$

Therefore also, by § 40, if u has a finite x derivative,

$$\frac{d}{dx}\sin u = \cos u \frac{du}{dx} \tag{1}$$

2. Replacing u by $\pi/2 - u$ in (1), we have

$$\frac{d}{dx}\cos u = \frac{d}{dx}\sin\left(\frac{\pi}{2} - u\right) = \cos\left(\frac{\pi}{2} - u\right)\frac{d}{dx}\left(\frac{\pi}{2} - u\right)$$

$$= -\sin u \frac{du}{dx} \tag{2}$$

3. The case of $\tan u$ will illustrate the method of finding the derivatives of the remaining functions. The results are tabulated in § 63.

$$\frac{d}{dx}\tan u = \frac{d}{dx}\frac{\sin u}{\cos u} = \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\frac{du}{dx} = \sec^2 u \frac{du}{dx}$$

63. Table of the derivatives of the trigonometric functions.

1. $\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$ 2. $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$

3. $\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$ 4. $\frac{d}{dx}\cot u = -\operatorname{cosec}^2 u \frac{du}{dx}$

5. $\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx}$

6. $\frac{d}{dx}\operatorname{cosec} u = -\operatorname{cosec} u \cot u \frac{du}{dx}$

Example. Find the derivatives of $\sin(1 - 2x)$, $\cos^2 x$, $\sec x^3$, $\tan^4 x^2$.

$$\frac{d}{dx}\sin(1 - 2x) = \cos(1 - 2x)\frac{d}{dx}(1 - 2x) = -2\cos(1 - 2x).$$

$$\frac{d}{dx}\cos^2 x = 2\cos x \frac{d}{dx}\cos x = -2\cos x \sin x = -\sin 2x.$$

$$\frac{d}{dx}\sec x^3 = \sec x^3 \tan x^3 \frac{d}{dx}x^3 = 3x^2 \sec x^3 \tan x^3.$$

$$\frac{d}{dx}\tan^4 x^2 = 4\tan^3 x^2 \cdot \frac{d}{dx}\tan x^2 = 4\tan^3 x^2 \sec^2 x^2 \cdot \frac{d}{dx}x^2$$

$$= 8x\tan^3 x^2 \sec^2 x^2.$$

EXERCISE IX

Differentiate the following functions 1–12.

1. $\sin 5x$, $\cos(x/3)$, $\tan(2x - 4)$, $\sec(2x/3)$, $\cot 6x$, $\operatorname{cosec} 9x$.
2. $\sin 2x^3$, $\cos\sqrt{x}$, $\tan x^{2/3}$, $\cot x^5$, $\sec(1/x)$, $\operatorname{cosec}(1/x^2)$.
3. $\sin^3 x$, $\sqrt{\cos x}$, $\tan^4 3x$, $\cot^3(1-x)$, $\sin^6 x^4$, $\sqrt{\cos x^2}$, $\tan^3(1/x)$.

4. $x \sin x + \cos x$ 5. $\tan x - x$ 6. $\sec^4 x - \tan^4 x$

7. $\cos^2 3x - \sin^2 3x$ 8. $\sqrt{1 - \cos x}$ 9. $\sin^2 x \cos^2 x$

10. $\sin \dfrac{2x}{1-x^2}$ 11. $\tan \sqrt{\dfrac{1+x}{1-x}}$ 12. $\sec \dfrac{1+x}{1-x}$

13. Show that $\dfrac{d}{dx}\sin x = \sin\left(x + \dfrac{\pi}{2}\right)$, $\dfrac{d^n}{dx^n}\sin x = \sin\left(x + n\dfrac{\pi}{2}\right)$; similarly, $\dfrac{d^n}{dx^n}\cos x = \cos\left(x + n\dfrac{\pi}{2}\right)$.

14. Find the maximum and minimum values of the following:

(1) $\sin x + \cos x$ (2) $2\sin x + \cos x$ (3) $3\sec x - \tan^2 x$

(4) $\sin x + \cos 2x$ (5) $2\sin x + \cos 2x$ (6) $2\sin x + \sin 2x$

15. Show that $\sin\theta/\theta$ continually decreases as θ increases from 0 to $\pi/2$.

16. A right circular cone whose vertical angle is 2θ is inscribed in a sphere of radius a. For what value of θ is its volume greatest? its curved surface?

17. The four sides of a quadrilateral are given. Show that the area of the quadrilateral is greatest when its opposite angles are supplementary.

18. Find when the area of a circular sector of given perimeter is greatest.

19. Show, by substitution, that the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has the parametric equations

$$x = a\cos\phi \qquad y = b\sin\phi$$

Show that the equation of the tangent at $x_1 = a\cos\phi_1$, $y_1 = b\sin\phi_1$ is

$$\frac{x}{a}\cos\phi_1 + \frac{y}{b}\sin\phi_1 = 1$$

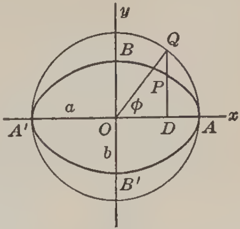

FIG. 35.

If P be any point of the ellipse and Q the point where DP produced meets the circle on the major axis $A'A$ as diameter, show that $\phi = DOQ$.

20. As ϕ varies, $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$ remains always tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If OX and OY denote its x- and y-intercepts, find when OXY is least; also when XY is least.

21. Show that for every rectilinear motion defined by an equation of the form $s = A \sin kt + B \cos kt$, the acceleration is proportional to $|s|$ and is directed toward O.

64. Inverses of $y = \sin x$, $\cos x$, $\tan x$. 1. If x, y be interpreted as the rectangular coordinates of a point, the graph of $y = \sin x$ between $x = 0$ and $x = \pi$ is the curve arc OCA, having the maximum point $C(\pi/2, 1)$, and symmetric with respect to the ordinate BC through this point. The graph between $x = -\pi$ and $x = 0$ is the arc $A'C'O$ symmetric to OCA with respect to O and having the minimum point $C'(-\pi/2, -1)$. Since $\sin(x + 2k\pi) = \sin x$, k denoting any integer, the complete graph is the wave curve got by repeating the double arc $A'C'OCA$ indefinitely to the left and right.

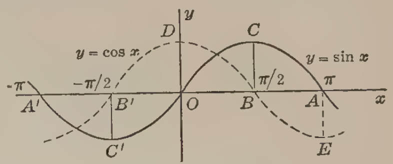

FIG. 36.

Between every two consecutive turning points of the graph, $y = \sin x$ is an increasing or decreasing function and therefore has a one valued inverse, § 52. The inverse between the points C' and C, that is, in the x-interval $(-\pi/2, \pi/2)$, is called the *principal inverse*, and is denoted by $x = \text{arc} \sin y$ or $x = \sin^{-1} y$. It is a continuous and increasing function of y, § 52.

2. Since $\cos x = \sin(x + \pi/2)$, the graph of $y = \cos x$ is the same as that of $y = \sin x$, shifted the distance $\pi/2$ to the left. It has a maximum point at $D(0, 1)$ and a minimum point at $E(\pi, -1)$. The inverse of $y = \cos x$ between D and E, that is, in the x-interval $(0, \pi)$, is called its *principal inverse* and is denoted by $x = \text{arc} \cos y$ or $x = \cos^{-1} y$. It is a continuous and decreasing function of y.

3. The graph of $y = \tan x$ between $x = -\pi/2$ and $x = \pi/2$ is the infinite curve branch AOC having the lines $x = -\pi/2$ and $x = \pi/2$ for asymptotes, and symmetric with respect to O. Since $\tan(x + k\pi) = \tan x$, k denoting any integer, the rest of the graph consists of repetitions of AOC to the left and right. Along each of these branches, $y = \tan x$ has a one valued inverse. The inverse along AOC, that is, in the x-interval $(-\pi/2, \pi/2)$, is called the *principal inverse* and is denoted by $x = \text{arc} \tan y$ or $x = \tan^{-1} y$. It is a continuous and increasing function of y.

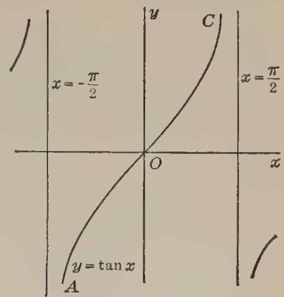

FIG. 37.

65. The functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$. We may interchange the rôles played by x and y in the preceding discussion. Thus $y = \sin^{-1} x$ is the inverse of $x = \sin y$ between $y = -\pi/2$ and $y = \pi/2$.

If in any equation $f(x, y) = 0$ (1) we interchange x and y, the equation becomes $f(y, x) = 0$ (2). If $P(a, b)$ be any point of the graph of (1), then $P'(b, a)$ is a point of the graph of (2). But it is readily seen from Fig. 38 that P' is symmetric to P with respect to the line $y = x$. Hence the graph of (2) is symmetric to the graph of (1) with respect to this line $y = x$. It therefore follows from the figures in § 64 that the graphs of the functions $y = \sin^{-1} x$, $y = \cos^{-1} x$, $y = \tan^{-1} x$ are as indicated below.

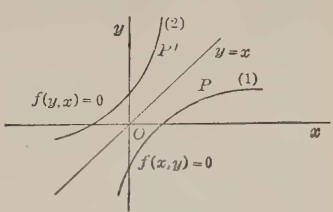

FIG. 38.

1. The functions $y = \sin^{-1} x$, $y = \cos^{-1} x$. These functions exist in the x-interval $(-1, 1)$ only. They are con-

tinuous in this interval. The graph of $y = \sin^{-1} x$ is the curve arc AOB: it shows the manner in which, when x increases from -1 to 1, y increases from $-\pi/2$ to $\pi/2$. The graph of $y = \cos^{-1} x$ is the curve arc CDE: it shows how, when x increases from -1 to 1, y decreases from π to 0. The arcs AOB and CDE are symmetric with respect to the line $y = \pi/4$. Hence $\sin^{-1} x + \cos^{-1} x = \pi/2$.

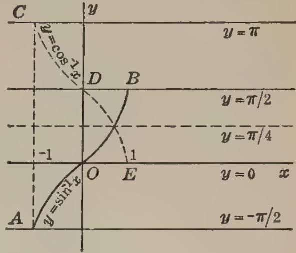

FIG. 39.

2. The function $y = \tan^{-1} x$. This function exists in the x-interval $(-\infty, \infty)$, and is continuous except at $x = \pm\infty$. Its graph is the infinite curve branch KOL: it shows how, when x increases from $-\infty$ to ∞, then y increases from $-\pi/2$ to $\pi/2$.

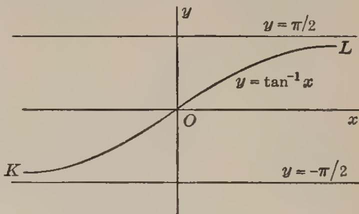

FIG. 40.

66. Derivatives of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$. 1. Since $y = \sin^{-1} x$ is the inverse of $x = \sin y$, $d(\sin^{-1} x)/dx$ is the reciprocal of $d(\sin y)/dy$, § 54. Hence

$$\frac{d}{dx}\sin^{-1} x = 1\Big/\frac{d}{dy}\sin y = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}} \tag{1}$$

For $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$, the radical having the $+$ sign because $\cos y$ is $+$ in the y-interval $(-\pi/2, \pi/2)$ to which $\sin^{-1} x$ belongs.

In like manner, we have

$$\frac{d}{dx}\cos^{-1} x = 1\Big/\frac{d}{dy}\cos y = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}} \tag{2}$$

$$\frac{d}{dx}\tan^{-1} x = 1\Big/\frac{d}{dy}\tan y = \frac{1}{\sec^2 y} = \frac{1}{x^2+1} \tag{3}$$

By § 40, we obtain from (1), (2), (3) the more general formulas:

$$1.\quad \frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$2.\quad \frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$3.\quad \frac{d}{dx}\tan^{-1}u = \frac{1}{u^2+1}\frac{du}{dx}$$

In 1. and 2., x is restricted to values for which $|u| < 1$.

We may also derive (1), (2), (3) as follows: The function $y = \sin^{-1}x$ is a solution of the equation $\sin y = x$, and, § 54, its derivative dy/dx is known to exist. Hence (by the reasoning in § 42), differentiating $\sin y = x$ with respect to x, we have $\cos y \cdot dy/dx = 1 \therefore dy/dx = 1/\cos y = 1/\sqrt{1-x^2}$.

EXERCISE X

Differentiate the following (1–20):

1. $\sin^{-1}x/a$ **2.** $\cos^{-1}3x$ **3.** $\tan^{-1}x/a$

4. $\sin^{-1}(2-3x)$ **5.** $\sin^{-1}x^2$ **6.** $\tan^{-1}1/x$

7. $\cos^{-1}1/x$ **8.** $\sin^{-1}1/x$ **9.** $\sin^{-1}(\cos x)$

10. $\tan^{-1}(\tan x)$ **11.** $\sin^{-1}(\sin 2x)$ **12.** $\cos^{-1}(1-x^2)^{1/2}$

13. $\cos^{-1}x^{1/2}$ **14.** $\sin^{-1}(1+2x)^{1/2}$ **15.** $\sin^{-1}[2(x-x^2)]^{1/2}$

16. $\sin^{-1}\sqrt{\sin x}$ **17.** $\tan^{-1}\dfrac{2x}{1-x^2}$ **18.** $\tan^{-1}\dfrac{1+x}{1-x}$

19. $\tan^{-1}\dfrac{x+a}{1-ax}$ **20.** $\tan^{-1}\left(\dfrac{1-\cos x}{1+\cos x}\right)^{1/2}$

21. Express the complete solution of the equation $\sin y = x$ in terms of $\sin^{-1}x$.

22. The points A and B are on Oy, 3 and 4 inches above O. At what points P on Ox is the angle APB the greatest?

23. We may define $\cot^{-1}x$, $\sec^{-1}x$, and $\operatorname{cosec}^{-1}x$ by the formulas:

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x \qquad \sec^{-1}x = \cos^{-1}\frac{1}{x} \qquad \operatorname{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$$

Find the graphs of these functions and the formulas for their derivatives.

VII. THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

67. Exponential and logarithmic functions. 1. Let a denote any number > 1. The symbol a^x is defined in elementary algebra for all rational values of x. Let c be an irrational number. It can be proved[1] that for all modes

[1] The proof is contained in the theorems that follow.

Theorem 1. *The function a^x increases as x increases through rational values.*

(1) If p, q are any positive integers, then $a^{p/q} > 1$. For $a > 1 \quad \therefore a^p > 1$ $\therefore \sqrt[q]{a^p} > 1$, that is, $a^{p/q} > 1$.

(2) Let r, s be any two rationals, $s > r$. Then $s - r > 0 \quad \therefore a^{s-r} > 1$ $\therefore a^{s-r}a^r > a^r \quad \therefore a^s > a^r$.

Theorem 2. *If $n \to \infty$ through integral values, then $a^n \to \infty$ and $a^{-n} \to 0$.*

For since $a > 1$, we may set $a = 1 + d$, where d is positive. Then $a^n = (1 + d)^n > 1 + nd$. But $\lim\limits_{n \to \infty} (1 + nd) = \infty$. Hence $\lim\limits_{n \to \infty} a^n = \infty$. Also $a^{-n} = 1/a^n$; hence $\lim\limits_{n \to \infty} a^{-n} = 0$.

Theorem 3. If $n \to \infty$ *through integral values, then $a^{1/n} \to 1$; therefore also $a^{-1/n} = 1/a^{1/n} \to 1$.*

For since $a^{1/n} > 1$, we may set $a^{1/n} = 1 + d_n$, where d_n is a positive number which varies with n. We then have $a = (1 + d_n)^n > 1 + nd_n$

$\therefore d_n < (a - 1)/n \quad \therefore \lim\limits_{n \to \infty} d_n = 0 \quad \therefore \lim\limits_{n \to \infty} a^{1/n} = 1.$

Theorem 4. *If c and x be rational, then $\lim\limits_{x \to c} a^x = a^c$.*

(1) Let $c = 0$. We can choose an integer n such that $-1/n < x < 1/n$, which makes $a^{-1/n} < a^x < a^{1/n}$. When $x \to 0$, then $n \to \infty$ and therefore $a^{1/n}, a^{-1/n} \to 1 \quad \therefore \lim\limits_{x \to 0} a^x = 1 = a^0$.

(2) Let $c \neq 0$. We have $a^x = a^c \cdot a^{x-c} \quad \therefore \lim\limits_{x \to c} a^x = a^c \lim\limits_{x \to c} a^{x-c} = a^c$.

Theorem 5. *If c be irrational and x rational, then a^x approaches a limit when $x \to c$.*

(1) Let $x \to c$ through increasing values and let g denote any rational $> c$. As x increases, a^x increases but remains less than a^g; it therefore approaches a limit $l (< a^g)$, § 5.

of approach of x to c as limit, through rational values, a^x will approach one and the same limit; we represent this limit by the symbol a^c. Thus $2^{\sqrt{3}} = \lim\limits_{x \to \sqrt{3}} 2^x$.

Having thus defined a^x for all real values of x, we can show that $y = a^x$ is a one valued, continuous and increasing function of x which increases from 0 to 1 as x increases from $-\infty$ to 0, and from 1 to ∞ as x increases from 0 to ∞. Its graph is a curve of the type (1) in Fig. 41, as may be illustrated by taking the case $y = 2^x$, plotting the points corresponding to $x = -3$, -2, -1, 0, 1, 2, 3, and drawing a smooth curve through these points.

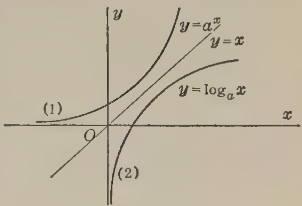

FIG. 41.

The function a^x is called an *exponential function of* x.

(2) Let $x' \to c$ through any rational values. Since $a^{x'} = a^x \cdot a^{x'-x}$, $\lim\limits_{x' \to c} a^{x'} = \lim\limits_{x \to c} a^x \lim\limits_{x, x' \to c} a^{x'-x} = l$.

Irrational exponents. The limit l which a^x approaches when $x \to c$ is denoted by a^c. Hence for c irrational and x rational we have $a^c = \lim\limits_{x \to c} a^x$.

Theorem 6. *The function a^x increases as x increases continuously.*

For let b and c be any two real numbers, $b < c$, and x any rational such that $b < x < c$. We have $a^b < a^x$ and $a^x < a^c$ $\quad \therefore a^b < a^c$.

Theorem 7. *The function a^x is continuous for every finite value c of x.*

For let $x \to c$ through any real values, and let x', x'' be rationals such that $x' < x < x''$. Then $a^{x'} < a^x < a^{x''}$. When $x', x'' \to c$, then $a^{x'}, a^{x''} \to a^c$ and therefore $a^x \to a^c$.

When $0 < a < 1$, a^x decreases as x increases through rational values. By modifying the preceding discussion to suit this circumstance it is easy to show that in this case also a^x is continuous for all finite values of x.

Theorem 8. *For any positive values of a, b and any real values of m, n, we have*

1. $a^m a^n = a^{m+n}$ $\qquad$ 2. $(a^m)^n = a^{mn}$ $\qquad$ 3. $(ab)^m = a^m b^m$

For let x, y be rationals which $\to m, n$. We have $a^x a^y = a^{x+y}$, $(a^x)^y = a^{xy}$, $(ab)^x = a^x b^x$. But when $x, y \to m, n$, then $a^x a^y = a^{x+y}$ gives $a^m a^n = a^{m+n}$ and $(ab)^x = a^x b^x$ gives $(ab)^m = a^m b^m$. Also when $x \to m$, $(a^x)^y = a^{xy}$ gives $(a^m)^y = a^{my}$, and this in turn, when $y \to n$, gives $(a^m)^n = a^{mn}$.

2. Since $y = a^x$ is a continuous increasing function of x in the x-interval $(-\infty, \infty)$, it has a continuous increasing inverse in that interval, § 52. This inverse is written $x = \log_a y$ and called the *logarithm of y to the base a*. It is a continuous increasing function of y in the y-interval $(0, \infty)$.

In like manner, $y = \log_a x$, which is the inverse of $x = a^y$, is a continuous increasing function of x in the x-interval $(0, \infty)$. Its graph is the curve (2) symmetric to (1) with respect to the line $y = x$. (See § 65.)

68. Properties of logarithms. 1. As is shown in the footnote, Th. 8, for a and b positive, and μ and ν real, we have $a^\mu a^\nu = a^{\mu+\nu}$, $(a^\mu)^\nu = a^{\mu\nu}$, $(ab)^\mu = a^\mu b^\mu$.

2. By § 67, 2., to every positive number m corresponds a real number μ such that $a^\mu = m$; we call μ the logarithm of m to the base a and write $\mu = \log_a m$: so that $m = a^{\log_a m}$.

Since $a^0 = 1$ and $a^1 = a$, we have $\log_a 1 = 0$ and $\log_a a = 1$. Since $a > 1$, $\log_a m$ is $+$ or $-$, according as $m \gtrless 1$. Since $\lim\limits_{x\to\infty} a^x = \infty$ and $\lim\limits_{x\to\infty} a^{-x} = 0$, we have $\log_a \infty = \infty$, and $\log_a 0 = -\infty$.

3. Suppose that $a^\mu = m$, $a^\nu = n$, so that $\mu = \log_a m$, $\nu = \log_a n$. Then

$$mn = a^\mu \cdot a^\nu = a^{\mu+\nu} \quad \text{that is,} \quad \log_a mn = \log_a m + \log_a n$$
$$m/n = a^\mu/a^\nu = a^{\mu-\nu} \quad \text{that is,} \quad \log_a m/n = \log_a m - \log_a n$$
$$m^r = (a^\mu)^r = a^{\mu r} \quad \text{that is,} \quad \log_a m^r = r \log_a m$$
$$\sqrt[s]{m} = \sqrt[s]{a^\mu} = a^{\mu/s} \quad \text{that is,} \quad \log_a \sqrt[s]{m} = [\log_a m]/s.$$

4. Suppose that $m = a^\mu = b^\nu$, so that $\mu = \log_a m$ and $\nu = \log_b m$.

Then $a^{\mu/\nu} = b$. Hence $\log_a b = \mu/\nu = \log_a m/\log_b m$, and therefore

$$\log_b m = \frac{\log_a m}{\log_a b}; \text{ also setting } m = a, \log_b a = \frac{1}{\log_a b}$$

Example. Express $\log_{10} \sqrt[3]{2000}\sqrt[4]{25/256}$ in terms of $\log_{10} 2$.

69. The number *e*. Consider the behavior of $(1 + 1/n)^n$ as n increases. Its values for $n = 1, 2, 3, 4$ are 2, 2.25, 2.37, 2.44. We shall prove that it continually increases with n, and approaches a number between 2 and 3 as limit when $n \to \infty$.

1. Let n take positive integral values only. The expansion of $(1 + 1/n)^n$ by the binomial theorem is

$$1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots + \frac{1}{n^n}$$

The sum of the first two terms is 2. The rest may be written

$$\frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots$$
$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right)$$

As n increases, the values of the first, second, $\cdots$ of these terms increase (for $1/n$, $2/n$, $\cdots$ decrease); the number of the terms also increases; hence the sum increases. But it remains less than

$$\frac{1}{2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{2 \cdot 3 \cdots n} < \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 1$$

Hence $(1 + 1/n)^n$ increases with n but remains less than $2 + 1$ or 3. It therefore approaches some number between 2 and 3 as limit, § 5. This number is denoted by e. It will be shown later that its value to the third decimal figure is 2.718. Hence, by definition,

$$e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = 2.718 \cdots$$

2. Let μ denote a variable which $\to \infty$ through positive real values of any kind, and n a variable integer such that

$$n \leqq \mu < n + 1$$

Then $$\left(1 + \frac{1}{n+1}\right)^n < \left(1 + \frac{1}{\mu}\right)^\mu < \left(1 + \frac{1}{n}\right)^{n+1}$$

The first and last of these expressions may be written

$$\left(1+\frac{1}{n+1}\right)^{n+1} \div \left(1+\frac{1}{n+1}\right) \text{ and } \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)$$

and therefore both of them $\rightarrow e$ when $n \rightarrow \infty$. Hence

$$\lim_{\mu\rightarrow\infty}\left(1+\frac{1}{\mu}\right)^{\mu} = e$$

3. Finally, we have

$$\left(1-\frac{1}{\mu}\right)^{-\mu} = \left(\frac{\mu}{\mu-1}\right)^{\mu} = \left(1+\frac{1}{\mu-1}\right)^{\mu}$$
$$= \left(1+\frac{1}{\mu-1}\right)^{\mu-1} \cdot \left(1+\frac{1}{\mu-1}\right)$$

Hence (by 2.),
$$\lim_{\mu\rightarrow\infty}\left(1-\frac{1}{\mu}\right)^{-\mu} = e$$

We have therefore proved that $(1 + 1/n)^n \rightarrow e$ when $n \rightarrow \infty$ or $-\infty$ through real numbers of whatever kind.

70. Natural logarithms. This name is given to logarithms to the base e. They are practically the only logarithms used in the Calculus. In what follows, $\log x$ will mean $\log_e x$.

71. Derivatives of $\log x$ and e^x. 1. If $y = \log x$, then

$$\Delta y = \log(x + \Delta x) - \log x = \log\frac{x+\Delta x}{x}$$
$$= \log\left(1+\frac{\Delta x}{x}\right) = \frac{\Delta x}{x}\log\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

Hence
$$\frac{\Delta y}{\Delta x} = \frac{1}{x}\log\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

Let $x/\Delta x = \mu$, and therefore $\Delta x/x = 1/\mu$. Then

$$\lim_{\Delta x\rightarrow 0}\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} = \lim_{\mu\rightarrow\infty}\left(1+\frac{1}{\mu}\right)^{\mu} = e$$

and since $\log x$ is continuous, when $(1 + 1/\mu)^\mu \to e$ then $\log (1 + 1/\mu)^\mu \to \log e = 1.$

Hence
$$\frac{d}{dx} \log x = \frac{1}{x} \tag{1}$$

2. Since $y = e^x$ is the inverse of $x = \log y$, we have, § 54, (1),

$$\frac{d}{dx} e^x = 1 \Big/ \frac{d}{dy} \log y = 1 \Big/ \frac{1}{y} = y = e^x$$

Hence
$$\frac{d}{dx} e^x = e^x \tag{2}$$

3. By § 40, we derive from (1) and (2) the more general formulas

$$\frac{d}{dx} \log u = \frac{1}{u}\frac{du}{dx} \qquad \frac{d}{dx} e^u = e^u \frac{du}{dx} \tag{3}$$

It is usually best to express the logarithm of a product in terms of the logarithms of the factors before differentiating it, and the like is true of the logarithm of a quotient, power, or root.

Example 1. $\dfrac{d}{dx} e^{-3x} = e^{-3x} \dfrac{d}{dx}(-3x) = -3e^{-3x}$

$$\frac{d}{dx} \log \sin x = \frac{1}{\sin x} \frac{d}{dx} \sin x = \frac{\cos x}{\sin x} = \cot x$$

$$\frac{d}{dx} \log \sqrt{\frac{(x^2+2)^3}{x^2+1}} = \frac{d}{dx} \frac{1}{2} [3 \log (x^2 + 2) - \log (x^2 + 1)]$$
$$= \frac{1}{2}\left[\frac{3 \cdot 2x}{x^2+2} - \frac{2x}{x^2+1}\right] = \frac{2x^3 + x}{(x^2+2)(x^2+1)}$$

Example 2. By § 68, 4., $\log_a x = \dfrac{\log x}{\log a} = \log x \cdot \log_a e.$

Hence
$$\frac{d}{dx} \log_a x = \frac{1}{x} \cdot \log_a e.$$

72. Logarithmic differentiation. It follows from § 67, 2. that a function u, when positive, may be expressed in the form $u = e^v$, where $v = \log u$. This gives

$$\frac{du}{dx} = \frac{d}{dx} e^v = e^v \frac{dv}{dx} = u \frac{dv}{dx} \tag{1}$$

Hence u has a derivative if its logarithm, v has one, and du/dx may be got by multiplying u by dv/dx. This process is called *logarithmic differentiation.*

EXAMPLE 1. Find the derivative of x^x.

Let $y = x^x$. Then $\log y = x \log x$. Hence, diff'g with respect to x, $\frac{1}{y}\frac{dy}{dx} = \log x + 1$. But $y = x^x$. Hence $\frac{dy}{dx} = x^x(\log x + 1)$.

EXAMPLE 2. If $y = a^u$, then $\log y = u \log a \quad \therefore \frac{1}{y}\frac{dy}{dx} = \frac{du}{dx}\log a$.

Hence $$\frac{d}{dx}a^u = a^u \log a \frac{du}{dx}.$$

EXAMPLE 3. If $y = u^n$, then $\log y = n \log u \quad \therefore \frac{1}{y}\frac{dy}{dx} = n\frac{1}{u}\frac{du}{dx}$.

Hence $$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

EXAMPLE 4. Find the derivative of $[x(1-x)^2/(x+1)^4]^{1/3}$ by logarithmic differentiation. Equating the expression to y, taking the logs of both members, diff'g with respect to x, we obtain finally

$$\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x} - \frac{2}{1-x} - \frac{4}{x+1}\right]\left[\frac{x(1-x)^2}{(x+1)^4}\right]^{1/3}$$

EXERCISE XI

Differentiate the following functions 1–20 with respect to x:

1. e^{4x}, e^{kx}, $\sqrt{e^x}$, $\sqrt[n]{e^x}$, $\sqrt{1/e^x}$, e^{3x^2}, $e^{\sin x}$, $e^{\tan 2x}$, $e^{\sin^{-1} x}$.

2. $\log 5x$, $\log(bx + c)$, $\log e^{3x}$, $\log(e^{x/2} + x)$, $\log^3 x$, $\log(\log x)$.

3. $\log(1 - 5x - 3x^4)^2$, $\log(\cos x)$, $\log(\tan 3x)$, $\log(\tan^{-1}x)$, $\log\sqrt{1 - 3x}$, $\log(1/x)$.

4. $\log\frac{x+a}{x-a}$ 5. $\log\sqrt{\frac{x+a}{x-a}}$ 6. $\log\sqrt{\frac{1+x^2}{1-x^2}}$ 7. $\log\sqrt{\frac{1+\cos x}{1-\cos x}}$

8. $\log(\sec x + \tan x)$ 9. $\log(x + \sqrt{x^2 + a^2})$ 10. $\log\sqrt{e^{2x} + x}$

11. $\frac{a}{2}(e^{x/a} - e^{-x/a})$ 12. $\log\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}$ 13. $\frac{1-e^x}{1+e^x}$

14. x^{x^n} 15. $(\sin x)^{\tan 2x}$ 16. $(1 - x)^x$

17. $x^{1/3}(1 - x)^{2/3}(1 - 2x^3)^{4/3}$

18. $\sqrt[3]{\dfrac{x^2+1}{x^5(1-x)^4}}$ **19.** $\dfrac{(2x-3)^2(1-5x)^4}{(x^3+1)(x^4+2)^2}$ **20.** $x^{\sqrt{2}} + x^{\sqrt[3]{3}}$

21. Prove that (1) $\dfrac{d^n}{dx^n} e^{kx} = k^n e^{kx}$

(2) $\dfrac{d^n}{dx^n} a^x = a^x(\log a)^n$ (3) $\dfrac{d^n}{dx^n} \log x = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$.

22. Find the turning points and graphs of (1) $y = xe^x$, (2) $y = x^2e^x$.

23. What are the characteristics of the motion of a point P on Ox if the equation be $x = e^{-t}\sin t$?

73. Hyperbolic functions. 1. In applied mathematics use is made of the functions $\sinh x$ and $\cosh x$ called *hyperbolic sine* and *hyperbolic cosine* and defined by the formulas

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \tag{1}$$

We also meet $\tanh x = \sinh x/\cosh x$, and so on.

2. Since, as shown[1] in Fig. 42, $y = \sinh x$ is a continuous increasing function of x between $x = -\infty$ and $x = \infty$, it has but one inverse, § 53. This inverse is denoted by $x = \sinh^{-1} y$. The equation

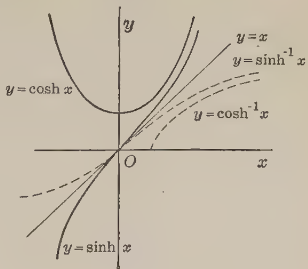

FIG. 42.

$$y = (e^x - e^{-x})/2$$

gives $$e^{2x} - 2ye^x - 1 = 0$$

Hence[2] $$e^x = y + \sqrt{y^2+1}$$

and $$x = \sinh^{-1} y = \log\,[y + \sqrt{y^2+1}] \tag{2}$$

The function $y = \cosh x$ decreases in the x-interval $(-\infty, 0)$ and increases in the interval $(0, \infty)$. It therefore

[1] The graph of $y = \cosh x$ is a *catenary*, the curve in which a uniform chain hangs under the action of gravity. If the length of its y-intercept in terms of some new unit be a, and the coordinates of any point in terms of this unit be x', y', the equation becomes $y' = a\cosh(x'/a)$.

[2] The solution $e^x = y - \sqrt{y^2+1}$ makes e^x negative and therefore x imaginary.

has two inverses. We select that belonging to $(0, \infty)$ as the principal inverse and denote it by $x = \cosh^{-1} y$.

Solving $\quad y = (e^x + e^{-x})/2 \qquad$ for x

gives[1]
$$x = \cosh^{-1} y = \log [y + \sqrt{y^2 - 1}] \tag{3}$$

The inverses of $x = \sinh y$ and $x = \cosh y$ are $y = \sinh^{-1} x$ and $y = \cosh^{-1} x$. The functions $\sinh^{-1} x$ and $\cosh^{-1} x$ are called the *inverse hyperbolic sine and cosine of x.* Interchanging x and y in (2) and (3), we get

$$\begin{aligned} \sinh^{-1} x &= \log [x + \sqrt{x^2 + 1}] \\ \cosh^{-1} x &= \log [x + \sqrt{x^2 - 1}] \end{aligned} \tag{4}$$

EXAMPLE 1. Prove the following properties of the hyperbolic functions:

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\operatorname{sech}^2 x + \tanh^2 x = 1$
3. $\dfrac{d}{dx} \sinh x = \cosh x$
4. $\dfrac{d}{dx} \cosh x = \sinh x$
5. $\dfrac{d}{dx} \sinh^{-1} \dfrac{x}{a} = \dfrac{1}{\sqrt{x^2 + a^2}}$
6. $\dfrac{d}{dx} \cosh^{-1} \dfrac{x}{a} = \dfrac{1}{\sqrt{x^2 - a^2}}$

EXAMPLE 2. Find the graph of $y = \tanh x$, showing that as x increases, from $-\infty$ to ∞, the curve rises from the asymptote $y = -1$ through O to the asymptote $y = 1$.

EXAMPLE 3. Define $\tanh^{-1} x$ and show that $\tanh^{-1} x = \dfrac{1}{2} \log \dfrac{1 + x}{1 - x}$.

[1] The solution $x = \log [y - \sqrt{y^2 - 1}] = - \log [y + \sqrt{y^2 - 1}]$ is the inverse belonging to the x-interval $(-\infty, 0)$.

VIII. CURVES IN POLAR COORDINATES

74. Polar coordinates. Take, as the elements of reference, the origin O and the direction Ox. Then, P being any point in the plane, join OP and represent the measures of xOP and OP by θ and r. We call θ, r the *polar coordinates* of P, referred to O, Ox.

Conversely, to find P when θ and r are given, we first draw from O the half-line which makes the angle θ with Ox, and then on this half-line when r is $+$, or on the half-line produced through O when r is $-$, lay off OP of length $|r|$. Thus $\theta = \pi/6$, $r = 3$ and $\theta = -5\pi/6$, $r = -3$ represent the same point P.

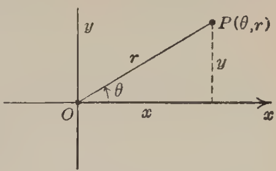

FIG. 43.

The relations between the polar coordinates θ, r of P and its x, y coordinates referred to the rectangular axes Ox, Oy in Fig. 43 are

$$x = r\cos\theta \qquad r = (x^2 + y^2)^{1/2}$$
$$y = r\sin\theta \qquad \tan\theta = y/x$$

75. Angle between tangent and radius vector. An equation of the form $r = f(\theta)$ represents some curve C; it is required to find the angle ψ made by the tangent at a point $P(\theta, r)$ of C with the radius vector OP. Let Q be the point $(\theta + \Delta\theta,\ r + \Delta r)$. With O as center draw the circular arc PR of radius r; also draw the chords PR and PQ.

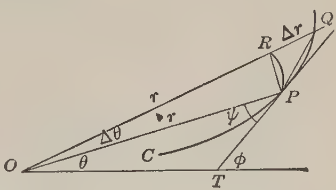

FIG. 44.

In the triangle PQR,

$$\frac{\sin Q}{\sin P} = \frac{PR}{RQ}.$$

Since the angle P is the supplement of $Q + R$, we have $\sin P = \sin (Q + R)$.

Let $\Delta\theta \to 0$. Then $Q \to \psi$ and $R \to \pi/2$ and therefore

$$\frac{\sin Q}{\sin P} = \frac{\sin Q}{\sin (Q + R)} \to \frac{\sin \psi}{\sin\left(\psi + \frac{\pi}{2}\right)} = \tan \psi$$

Also $RQ = \Delta r$, and arc $PR = r\,\Delta\theta$. Hence, when $\Delta\theta \to 0$,

$$\frac{PR}{RQ} = \frac{PR}{\text{arc } PR}\,\frac{r\,\Delta\theta}{\Delta r} \to r \bigg/ \frac{dr}{d\theta}$$

Therefore
$$\tan \psi = r \bigg/ \frac{dr}{d\theta} \tag{1}$$

The figure and proof correspond to the case in which r increases with θ. When r decreases, $dr/d\theta$ is $-$ and (1) gives the obtuse angle made by the tangent with OP. In both cases ψ is the angle between OP and the half-tangent PT which, when θ increases, follows P in its motion along C. The proof fails when $dr/d\theta$ is 0; but (1) continues to hold good since ψ is then $\pi/2$.

Observe in Fig. 44 that θ, ψ, and the slope angle ϕ satisfy

$$\phi = \theta + \psi \tag{2}$$

Example 1. For the cardioid $r = a(1 - \cos \theta)$, show that $\psi = \theta/2$.

$$\tan \psi = r \bigg/ \frac{dr}{d\theta} = \frac{1 - \cos \theta}{\sin \theta} = \tan \frac{\theta}{2}$$

$$\therefore \quad \psi = \frac{\theta}{2}.$$

To $\theta = 0, \frac{\pi}{2}, \pi$ correspond the points O, $A\left(\frac{\pi}{2}, a\right)$, $B(\pi, 2a)$.

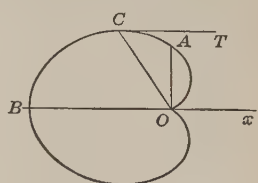

Fig. 45.

At the highest point C, we have $xOC + OCT = \pi$. But $xOC = 2\,OCT$ $\therefore xOC = \frac{2}{3}\pi$ $\therefore OC = \frac{3}{2}a$.

Example 2. Find ψ at the point of $r = a \sin^3 (\theta/3)$ where $\theta = \pi$.

76. Graphs of equations $r = f(\theta)$. To find the graph C of an equation of the form $r = f(\theta)$, we compute r for significant values of θ and pass a smooth curve through the points thus determined. If $f(-\theta) = f(\theta)$, the curve is symmetric with respect to Ox (§ 75, Ex. 1). To find whether C passes through O, we may set $f(\theta) = 0$; its real roots, if any, are the angles at which C cuts Ox at O. The sign of $dr/d\theta$ indicates when r increases with θ and when r decreases. We can find the points where the slope is ∞ and 0 by solving $dx/d\theta = 0$ and $dy/d\theta = 0$ for θ, or by aid of the relation $\phi = \theta + \psi$.

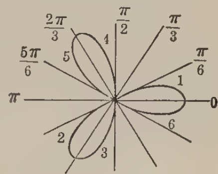

FIG. 46.

EXAMPLE 1. Find the graph of $r = a \cos 3\theta$.

θ	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$	π
3θ	0	$\pi/2$	π	$3\pi/2$	2π	$5\pi/2$	3π
r	a	0	$-a$	0	a	0	$-a$

As θ increases from 0 to π we trace successively the arcs 1, 2, 3, 4, 5, 6.

EXAMPLE 2. The curve $r^2 = a^2 \cos 2\theta$ is called a lemniscate. Show that it has the shape indicated in Fig. 47. Show that for it

$$\psi = 2\theta + \pi/2.$$

Find its x, y equation.

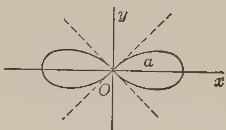

FIG. 47.

EXAMPLE 3. The curve $r = e^{k\theta}$, $k > 0$, is called a logarithmic spiral. Show that it has the shape indicated in Fig. 48, and that when $\theta \to -\infty$ the tracing point approaches 0 asymptotically. Show also that $\tan \psi$ has the constant value $1/k$.

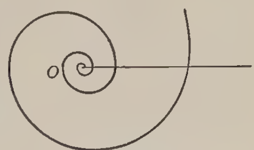

FIG. 48.

EXERCISE XII

1. Show that $r = a \cos \theta$ is a circle of diameter a, touching Oy at O; also that $\psi = \theta + \pi/2$.

2. Show that $r = a \sin \theta$ is a circle of diameter a, touching Ox at O; also that $\psi = \theta$.

3. Prove that for any curve $r = a(\sin k\theta)^{1/k}$, $\psi = k\theta$. Draw the curve for the case $k = 2$.

4. Trace the curve $r = a \sin 3\theta$ and find $\tan \psi$ when $\theta = \pi/3$.

5. Trace the curve $r = a \cos 2\theta$, showing that it has four loops. How many loops has $r = \cos n\theta$?

6. The curve $r = l/(1 - e \cos \theta)$ is a conic of eccentricity e with a focus at O and with its axis on Ox; trace it for the case $e = 1$, showing that $\psi = \pi - \theta/2$.

7. Trace the spiral of Archimedes $r = a\theta$. Find ψ.

8. Trace the limaçon $r = c - a \cos \theta$ for the cases $c = 2a$ and $c = a/2$.

9. Trace the curves (1) $r = a \sin^3 (\theta/3)$ (2) $r = \cos \theta + \sin \theta$.

10. Find the points where the slope of $r = a \cos 2\theta$ is 0.

11. Find the points where the slope of the curve $r = a(1 - \cos \theta)$ is ∞.

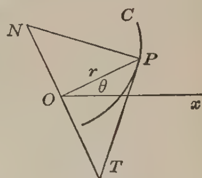

FIG. 49.

12. In Fig. 49, PT and PN are the tangent and normal to C at P, and TON is at right angles to $OP = r$. Prove that $OT = r^2 \Big/ \dfrac{dr}{d\theta}$, $ON = dr/d\theta$. If $f(\theta)$ is ∞ for $\theta = \theta_1$, then C has an infinitely distant point $P_\infty(\theta_1, \infty)$, the lines OP_∞ and TP_∞ are parallel, and TP_∞ is an asymptote to C. Find the asymptotes of the curve $r = a \tan \theta$; also trace the curve.

13. Show that $r \cos(\theta - \alpha) = p$ is the equation of a line whose perpendicular distance from O is p, α being the angle which the perpendicular makes with Ox.

IX. CURVATURE

77. Differential of arc. Let s denote the length of the arc of the curve S between the fixed point A and the variable point $P(x, y)$. Evidently s is a function of the x of P; its derivative ds/dx may be found as follows:

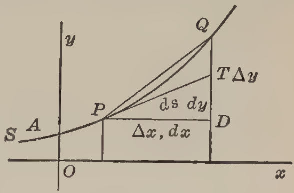

FIG. 50.

Let Q be the point $(x + \Delta x, y + \Delta y)$, and Δs the length of the arc PQ. The length of PQ is $[(\Delta x)^2 + (\Delta y)^2]^{1/2}$; hence

$$\frac{\Delta s}{\Delta x} = \frac{\Delta s}{PQ}\frac{PQ}{\Delta x} = \frac{\Delta s}{PQ}\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]^{1/2}$$

Let $\Delta x \to 0$. Then $\Delta s/PQ \to 1$, and therefore

$$\frac{ds}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} \tag{1}$$

Multiply both members of (1) by dx; we get, § 34,

$$ds = [(dx)^2 + (dy)^2]^{1/2} \tag{2}$$

Since $dx = PD$ and $dy = DT$, (2) shows that $ds = PT$. Since dx, dy, ds are the sides of the triangle PDT and DPT is the slope angle ϕ,

$$dx = \cos\phi\, ds \qquad dy = \sin\phi\, ds \quad (3)$$

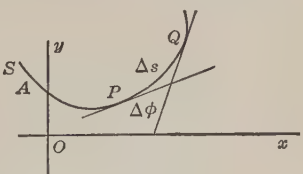

FIG. 51.

78. Curvature. Let PQ be a convex arc, § 57, of a given curve S. Let Δs be the length of the arc PQ, and $\Delta\phi$ the circular measure of the angle which the tangent at Q makes with that at P. The ratio $\Delta\phi/\Delta s$ is called the *mean curvature* of the arc

PQ, expressed in radians per unit of arc. When Q is made to move along S into coincidence with P, $\Delta\phi/\Delta s$ approaches the limit $d\phi/ds$. This limit is called the *curvature of S at P*, and is denoted by K. Hence, by definition,

$$K = \frac{d\phi}{ds} \qquad (1)$$

We may interpret s as the length of the arc from any fixed point A on S to P, ϕ as the angle which the tangent at P makes with Ox, and $K = d\phi/ds$ as the rate of change of ϕ with respect to s (in radians per unit of arc).

When no convention is made as to the positive sense of motion along S, the sign of K is ignored and $|K|$ is called the curvature of S at P.

The curvature of a circle is the same at every point and is the reciprocal of the radius.

For since the angles PDE and PCQ are equal, we have $\Delta s = r\,\Delta\phi$. Hence

$$\frac{\Delta\phi}{\Delta s} = \frac{1}{r} \quad \text{and therefore} \quad \frac{d\phi}{ds} = \frac{1}{r}$$

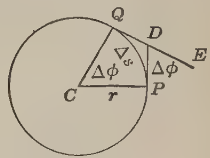

FIG. 52.

79. Radius of curvature. The reciprocal of the curvature K is called the *radius of curvature* of S at P, and is denoted by R. Hence, by definition

$$R = \frac{ds}{d\phi} \qquad (1)$$

When the equation of S is given in rectangular coordinates, R may be found as follows: Let ϕ be the circular measure of the positive or negative acute angle which the tangent at P makes with Ox; then $-\pi/2 < \phi < \pi/2$, and therefore, § 65, $\phi = \tan^{-1}(dy/dx)$.

We have, § 55 (3), $R = \dfrac{ds}{d\phi} = \dfrac{ds}{dx}\Big/\dfrac{d\phi}{dx}$

But, § 77 (1), $$\frac{ds}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

and $$\frac{d\phi}{dx} = \frac{d}{dx}\tan^{-1}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}\Big/\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

Hence $$R = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}\Big/\frac{d^2y}{dx^2} \tag{2}$$

The reciprocal of the fraction (2) is the curvature K at P. It is positive or negative according as d^2y/dx^2 is + or − at P, that is, according as S is concave upward or downward. At a point P where dy/dx is 0, we have $K = d^2y/dx^2$.

EXAMPLE 1. Find R at the point $(2, -1)$ on the curve $x^3 + y^3 = 7$.

$$x^2 + y^2\frac{dy}{dx} = 0 \qquad 2x + 2y\left(\frac{dy}{dx}\right)^2 + y^2\frac{d^2y}{dx^2} = 0$$

Substituting $x, y = 2, -1$, we find

$$\frac{dy}{dx} = -4, \quad \frac{d^2y}{dx^2} = 28, \quad \therefore \text{ by (2), } R = 17^{3/2}/28.$$

EXAMPLE 2. A locomotive is running on a track which has the shape of the parabola $y = x^2$, a mile being the unit of length. At what rate per mile is its direction changing when $x = 1$?

Here $dy/dx = 2x$, $d^2y/dx^2 = 2$. $\therefore K = 2/(1 + 4x^2)^{3/2}$. Hence when $x = 1$, $K = 2/5^{3/2}$ radians/mile.

EXAMPLE 3. Find the following radii of curvature.

1. $y = 1/x^2$ at $x = -1$
2. $y = \sin x$ at $x = \pi/2$
3. $y = \tan x$ at $x = \pi/4$
4. $y = e^x$ at $x = 0$
5. $y^2 - y + x = 0$ at $(0, 0)$
6. $y^2 - 2xy + x = 0$ at $(4/3, 2)$
7. $x^3 + y^3 = 3xy$ at $(3/2, 3/2)$

80. Center of curvature. On the normal to S at P and to the concave side of S, take PC equal to the radius of curvature R at P. Then with C as center and CP as radius describe the circle S_1. We call S_1 the

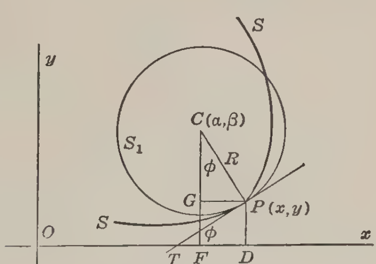

FIG. 53.

circle of curvature, and C the *center of curvature* of S at P. The circle S_1 touches (and ordinarily crosses) S at P. Its curvature $1/R$ is that of S at P.

Let the coordinates of P and C be (x, y) and (α, β). We can find α, β in terms of x, y as follows:

$$\text{Since} \qquad GCP = DTP = \phi$$
$$\alpha = OF = OD - GP = x - R \sin \phi$$
$$\beta = FC = DP + GC = y + R \cos \phi \tag{3}$$

Hence, expressing $\cos \phi$ and $\sin \phi$ in terms of $\tan \phi = dy/dx$ and using the formula for R, § 79 (2),

$$\alpha = x - \frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right] \Big/ \frac{d^2y}{dx^2}, \quad \beta = y + \left[1 + \left(\frac{dy}{dx}\right)^2\right] \Big/ \frac{d^2y}{dx^2} \tag{4}$$

These formulas (3), (4) will be found true for all cases of the figure if it be remembered that since $-\pi/2 < \phi < \pi/2$, $\cos \phi$ is always $+$ and $\operatorname{sgn} \sin \phi = \operatorname{sgn}(dy/dx)$, and that, when S is concave downward, d^2y/dx^2 is $-$.

EXAMPLE. For $y = x^3$ at $(1, 1)$ we have $dy/dx = 3x^2 = 3$, $d^2y/dx^2 = 6x = 6$. Hence, by (2) and (4), $\alpha = -4$, $\beta = 8/3$, $R = 10^{3/2}/6$, and the circle of curvature is $(x + 4)^2 + (y - 8/3)^2 = 10^3/6^2$.

81. The evolute. When P moves along S, C traces a curve S'. This curve S' is called the *evolute* of S. If the equation of S be given in the form $y = f(x)$, we can express the second members of (4) in terms of the x of P; the equations (4) then become parametric equations of S' in terms of the parameter x.

The normal PC to S at P is the tangent to S' at C. And if P_1, P_2 be two points of S, and C_1, C_2 the corresponding points of S', and if PC increases only or decreases only as P moves from P_1 to P_2, then arc $C_1C_2 = |P_2C_2 - P_1C_1|$.

For, differentiating the equations (3) with respect to x, we get

$$\frac{d\alpha}{dx} = 1 - \frac{dR}{dx}\sin\phi - R\cos\phi\frac{d\phi}{dx} \qquad \frac{d\beta}{dx} = \frac{dy}{dx} + \frac{dR}{dx}\cos\phi - R\sin\phi\frac{d\phi}{dx}$$

But, § 77 (3), § 79 (1),

$$R\cos\phi\frac{d\phi}{dx} = \frac{ds}{d\phi}\frac{dx}{ds}\frac{d\phi}{dx} = 1, \qquad R\sin\phi\frac{d\phi}{dx} = \frac{ds}{d\phi}\frac{dy}{ds}\frac{d\phi}{dx} = \frac{dy}{dx}.$$

Hence $\qquad \dfrac{d\alpha}{dx} = -\dfrac{dR}{dx}\sin\phi\ (a) \qquad \dfrac{d\beta}{dx} = \dfrac{dR}{dx}\cos\phi\ (b)$

1. Dividing (b) by (a) gives $\dfrac{d\beta}{d\alpha} = -\cot\phi = -1\Big/\dfrac{dy}{dx}$, which shows that PC touches S' at C.

2. Squaring (a) and (b) and adding gives $\left(\dfrac{d\alpha}{dx}\right)^2 + \left(\dfrac{d\beta}{dx}\right)^2 = \left(\dfrac{dR}{dx}\right)^2$

Therefore $(ds')^2 = (dR)^2$, where ds' is the differential of arc of S', § 77 (2). Hence if $R = PC$ increases as C moves from C_1 to C_2, and s' is the length of the arc C_1C, we have $ds' = dR$ and therefore $s' = R + c$, where c is a constant, § 104. When C is at C_1 this equation becomes $0 = P_1C_1 + c$, and when C is at C_2 it becomes arc $C_1C_2 = P_2C_2 + c$; hence, eliminating c, arc $C_1C_2 = P_2C_2 - P_1C_1$.

If R decreases as C moves from C_1 to C_2, it increases as C moves from C_2 to C_1; hence arc $C_2C_1 = P_1C_1 - P_2C_2$.

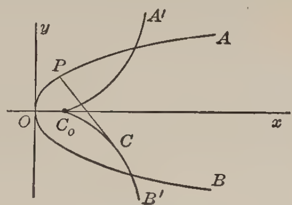

FIG. 54.

EXAMPLE. Find the evolute of the parabola $y^2 = 4x$.

Substituting $y = 2x^{1/2}$ (OA) in (4) gives

$\alpha = 2 + 3x$, $\beta = -2x^{3/2}$ (C_0B')

Substituting $y = -2x^{1/2}$ (OB) in (4) gives

$\alpha = 2 + 3x$, $\beta = 2x^{3/2}$ (C_0A')

Eliminating the parameter x, gives $\beta^2 = \frac{4}{27}(\alpha - 2)^3$ ($A'C_0B'$)

$$\text{Arc } C_0C = PC - OC_0 = PC - 2.$$

EXERCISE XIII

1. Show that if $x = \phi(t)$, $y = \psi(t)$, then $ds = [\phi'^2(t) + \psi'^2(t)]^{1/2}\,dt$.

2. For the ellipse $x = a\cos\theta$, $y = b\sin\theta$, p. 63, Ex. 19, show that $ds = a[1 - e^2\cos^2\theta]^{1/2}\,d\theta$, where e denotes the eccentricity, so that $a^2e^2 = a^2 - b^2$.

3. For the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, § 94, show that $ds = 2a\sin(\theta/2)\,d\theta$.

4. For the parabola $y^2 = 4\,ax$ find ds in terms of x and dx, also in terms of y and dy.

5. Show that the curvature at a point of inflection is 0. What is the curvature of a straight line?

6. Draw the parabola $y^2 = 4\,x$ and its circle of curvature at the point where $y = 2$.

7. Find the following circles of curvature:

1. $y = \cos x$ where $x = \pi$ 2. $y^2 + xy - 15 = 0$ at $(2, 3)$
3. $y^3 + x - 2\,y = 0$ at $(1, 1)$

8. For the equilateral hyperbola $xy = a^2$, show that
$$R = (x^2 + y^2)^{3/2}/2\,a^2$$

9. For the ellipse $x^2/a^2 + y^2/b^2 = 1$, show that
$$R = (a^4y^2 + b^4x^2)^{3/2}/a^4b^4 = (a^2 - e^2x^2)^{3/2}/ab$$

10. At what points of the curve $y = x^3$ is $|R|$ a minimum?

11. Show that $\left[1 + \left(\dfrac{dx}{dy}\right)^2\right]^{3/2} \Big/ \dfrac{d^2x}{dy^2} = -\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2} \Big/ \dfrac{d^2y}{dx^2}$.

12. By aid of Ex. 11, find R for the curve $x = y + y^2 + y^3$ at the point where $y = -1$.

13. For the curve $x = \phi(t)$, $y = \psi(t)$, show that
$$R = [\phi'^2(t) + \psi'^2(t)]^{3/2}/[\phi'(t)\psi''(t) - \phi''(t)\psi'(t)]$$

14. Show for the cycloid (Ex. 3) that $R = -4\,a \sin(\theta/2)$

15. For a curve $r = f(\theta)$ in polar coordinates, show that
$$R = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2} \Big/ \left[r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right]$$

16. Find R for the curves $r = e^{k\theta}$ and $r = a\,(1 - \cos\theta)$.

17. Show that the circle $(x - a)^2 + (y - b)^2 = r^2$ which passes through the point $P(x, y)$ of the curve S and has at P the same values of dy/dx and d^2y/dx^2 that S has is the circle of curvature of S at P.

18. Show that the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$ has the equation $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

19. Let N be the point where the normal to the curve S at the point P is met by the normal at Q. We have $PN = PQ \sin PQN/\sin PNQ$. Show that $\lim\limits_{Q \to P} PN = R$.

20. A point P is moving away from O on the upper half of the parabola $y^2 = x$. Its rate of motion along the curve is constant, 2 ft./sec. How fast is its direction changing, and how fast are its projections on Ox and Oy moving, when its x is 3 ft.?

X. CURVILINEAR MOTION

82. Vectors. The displacement of a point from the position A to the position B may be represented by the directed line segment AB. Such a directed line segment is called a *vector*. In this chapter AB will mean vector AB.

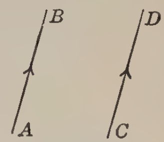

FIG. 55.

Two vectors AB and CD are *equal* when and only when they have the same length and direction.

If k be any real number, kAB denotes the vector whose length is $|k|$ times the length of AB and which has the same direction as AB or the opposite direction according as k is positive or negative. In particular,

$$-AB = BA \tag{1}$$

83. Vector addition. The displacement of a point from A to B, followed by its displacement from B to C is equivalent to the single displacement from A to C. Corresponding to this, the vector AC is called the *sum* of the vectors AB and BC.

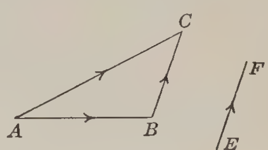

FIG. 56.

Hence, by definition,

$$AB + BC = AC \tag{2}$$

To add EF to AB, take $BC = EF$ and then apply (2).

If a, b, c are vectors, and k any real number, then (as with numbers)

1. $a + b = b + a$ **2.** $(a + b) + c = a + (b + c)$

3. $k(a + b) = ka + kb$

The proofs of 1. and 3. are as follows; that of 2. is left to the reader.

$$\begin{aligned} a + b &= AB + BC \\ &= AC \\ &= AD + DC \\ &= b + a \end{aligned}$$

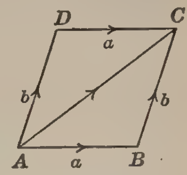

FIG. 57.

Take $A'B = kAB$, $BC' = kBC$

Then $ka + kb = A'B + BC'$

$$= A'C' = kAC$$
$$= k(a + b)$$

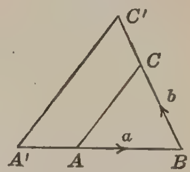

FIG. 58.

We define $AC - AB$ (Fig. 56) as $AC + (-AB)$ or as $AC + BA = BA + AC = BC$. Hence

$$AC - AB = BC \qquad (3)$$

84. Vector components. The relation $AC = AB + BC$ is also read: AB and BC are *components* of AC, and AC is the *resultant* of AB and BC. When AB, BC are parallel to Ox, Oy, they are called the x- and *y-components* of AC.

85. Velocity. A point P is supposed to be moving on a curve C. Let P and P' be its positions at the instants t and $t + \Delta t$, and let Δs denote the length of the arc PP'. Produce the vector PP' to a point U such that

$$\frac{PP'}{\Delta t} = PU$$

When $\Delta t \to 0$, PU will approach as limit a vector PV tangent to C at P. This vector

$$PV = \lim_{\Delta t \to 0} \frac{PP'}{\Delta t} \qquad (1)$$

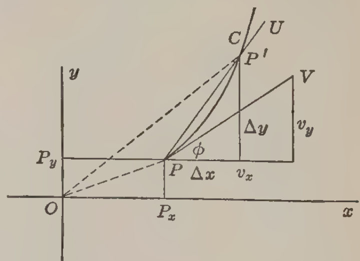

FIG. 59.

is called the *velocity* of the moving point P at the instant t; for it represents in magnitude and direction the time rate of displacement of P at the instant t.

Let v denote the length of PV, and $\overline{PP'}$ the length of PP'; then $\lim\limits_{\Delta t \to 0} \overline{PP'}/\Delta t = \lim\limits_{\Delta t \to 0} \Delta s/\Delta t$, and therefore

$$v = \frac{ds}{dt} \tag{2}$$

Let the coordinates of P be x, y, and those of P', $x + \Delta x$, $y + \Delta y$. Then, § 83, 3.,

$$\frac{PP'}{\Delta t} \text{ has the } x\text{- and } y\text{-components } \frac{\Delta x}{\Delta t} \text{ and } \frac{\Delta y}{\Delta t}$$

Hence the x- and y-components of PV, denoted by v_x and v_y, are

$$v_x = \frac{dx}{dt} \qquad \text{and} \qquad v_y = \frac{dy}{dt} \tag{3}$$

the same, therefore, as the velocities of P_x and P_y, the projections of P on Ox and Oy.

If ϕ is the angle which PV makes with the direction Ox,

$$v_x = v \cos \phi \qquad v_y = v \sin \phi$$
$$v = (v_x^2 + v_y^2)^{1/2} \qquad \tan \phi = v_y/v_x \tag{4}$$

Evidently PV is determined when v and ϕ are known. We find v and ϕ by substituting in (4) the values of v_x, v_y obtained by (3).[1]

We call v the *speed* and ϕ the *direction angle* of the motion at P.

Example. If the equations of C in terms of t are $x = t^2$, $y = 2t$, then $v_x = 2t$, $v_y = 2$; hence at $t = 2$ we have $v_x = 4$, $v_y = 2$ $\therefore v = 2\sqrt{5}$ and $\tan \phi = 1/2$; also v_x, v_y are $+$ $\therefore 0 < \phi < \pi/2$.

86. Derivative of a vector. We may call OP the *position vector* of P referred to O. We have $PP' = OP' - OP$,

[1] The signs of v_x and v_y indicate which of the angles given by $\tan \phi = v_y/v_x$ to take as the ϕ of PV.

§ 83 (3), so that PP' is the increment of OP corresponding to the t-increment Δt and may be represented by $\Delta(OP)$. Hence if we extend to vectors the notion of derivative as defined in § 23, we have from § 85 (1)

$$PV = \lim_{\Delta t \to 0} \frac{\Delta(OP)}{\Delta t} = \frac{d}{dt} OP \tag{5}$$

The velocity at P is the t-derivative of the position vector of P.

87. Acceleration. 1. As in the case of rectilinear motion, the time rate of change of velocity is called *acceleration.* Hence by definition

$$\textit{Acceleration at } P = \frac{d}{dt}(PV) \tag{6}$$

Take $O_1\bar{x}$, $O_1\bar{y}$ parallel to Ox, Oy in Fig. 59, also $O_1Q = PV$. The coordinates of Q are $\bar{x} = v_x$, $\bar{y} = v_y$. As the point P in Fig. 59 traces the curve C, the point Q traces a curve C', which is called the *hodograph* or *velocity curve* of P.

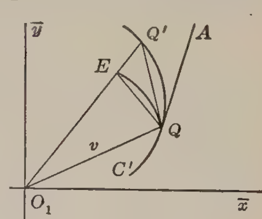

FIG. 60.

Since $d(PV)/dt = d(O_1Q)/dt$, it follows from §§ 85, 86, that the acceleration at P is a vector QA tangent to the curve C' at Q and having the $\bar{x}$-, $\bar{y}$-, or x-, y-components

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \tag{7}$$

Let a denote the length of QA, and ϕ' the angle which QA makes with $O_1\bar{x}$ or Ox; then

$$a_x = a \cos \phi' \qquad a_y = a \sin \phi'$$
$$a = (a_x^2 + a_y^2)^{1/2} \qquad \tan \phi' = a_y/a_x \tag{8}$$

Evidently the acceleration QA is determined when its *magnitude* a and its *direction angle* ϕ' are known; we find a and ϕ' by substituting in (8) the values of a_x, a_y given by (7).[1]

[1] The signs of a_x and a_y indicate which of the angles given by $\tan \phi' = a_y/a_x$ is the ϕ' of QA.

2. Let a_T and a_N denote the lengths of the components of QA in the directions of the tangent and normal to the curve C at P, that is, parallel and perpendicular to O_1Q. They can be found as follows:

Let O_1Q' correspond to $t + \Delta t$. With O_1 as center and v as radius draw the arc QE. We have $QQ' = QE + EQ'$; hence

$$\frac{QQ'}{\Delta t} \text{ has the components } \frac{EQ'}{\Delta t} \text{ and } \frac{QE}{\Delta t}$$

The directions approached by these components when $\Delta t \to 0$ are those of the tangent and normal to C at P. Also the length of EQ' is Δv and that of arc QE is $v\Delta\phi$. Hence

$$a_T = \frac{dv}{dt} \qquad a_N = v\frac{d\phi}{dt} \tag{9}$$

Again if R be the radius of curvature of C at P, we have, by § 79 (1),

$$v\frac{d\phi}{dt} = v\frac{d\phi}{ds}\frac{ds}{dt} = \frac{v^2}{R}; \text{ hence } \qquad a_N = \frac{v^2}{R} \tag{10}$$

In the case of rectilinear motion, and in that case only, a_N is 0 at every instant (since $d\phi/dt$ is then 0), and a_T is the entire acceleration.

When P describes equal arcs of C in equal intervals of time, its speed $v = ds/dt$ is constant, hence $dv/dt = 0$ and therefore a_N is the entire acceleration.

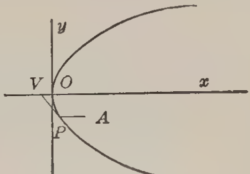

Fig. 61.

Example 1. For the motion $x = t^2/2$, $y = t$, (clockwise on $y^2 = 2x$), we have $v_x = t$, $v_y = 1$, $v = (t^2+1)^{1/2}$, $\tan\phi = 1/t$, $a_x = 1$, $a_y = 0$, $a = 1$, $\tan\phi' = 0$, $a_T = t/(t^2+1)^{1/2}$, $a_N = -1/(t^2+1)^{1/2}$.

The hodograph is $y = 1$.

At $t = -1$, we have (Fig. 61)

$$v = \sqrt{2}, \quad \phi = 3\pi/4 \ (PV); \quad \text{also } a = 1, \ \phi' = 0 \ (PA).$$

EXAMPLE 2. A point P is moving away from O on the upper half of $y^2 = 4x$ with the constant speed 10. What is the acceleration at (4, 4)?

We have $\quad y^2 - 4x = 0 \quad (1) \qquad y\dfrac{dy}{dt} - 2\dfrac{dx}{dt} = 0 \qquad (2)$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 100 \qquad (3)$$

Differentiating (2), $\left(\dfrac{dy}{dt}\right)^2 + y\dfrac{d^2y}{dt^2} - 2\dfrac{d^2x}{dt^2} = 0 \qquad (4)$

Differentiating (3), $\dfrac{dx}{dt}\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt}\dfrac{d^2y}{dt^2} = 0 \qquad (5)$

Setting $x, y = 4, 4$ in (2), (3) gives $\dfrac{dx}{dt} = 4\sqrt{5}$, $\dfrac{dy}{dt} = 2\sqrt{5}$. Hence, by (4), (5), we have $a_x = 2$, $a_y = -4$, $a = 2\sqrt{5}$.

88. Rotation. When the path of P is a circle about O as center, the motion is called a *rotation about O*. Let r be the radius, and θ the circular measure of xOP. Evidently the motion is completely determined when θ is a known function of t.

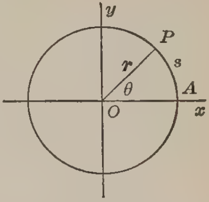

FIG. 62.

$\dfrac{d\theta}{dt}$ is the *angular velocity:* ω

$\dfrac{d^2\theta}{dt^2}$ is the *angular acceleration:* α

The length of the arc AP is $r\theta$, so that $ds/dt = r\,d\theta/dt$. The direction angle ϕ of the velocity vector of P is $\theta + \pi/2$, so that $d\phi/dt = d\theta/dt$. Hence for the motion of P

$$v = r\frac{d\theta}{dt} \qquad a_T = r\frac{d^2\theta}{dt^2} \qquad a_N = r\left(\frac{d\theta}{dt}\right)^2 \qquad (11)$$

We may obtain v_x, v_y, a_x, a_y from $x = r\cos\theta$, $y = r\sin\theta$.

EXAMPLE. A fly wheel of 4 ft. radius is making 150 revolutions a minute. (1) For a point P on its rim find v, a_T, a_N in feet per sec. (2) What constant angular acceleration would bring the wheel from rest to its present angular velocity in 1 min.?

(1) Since 1 revolution $= 2\pi$ radians, $\omega = d\theta/dt = 150 \cdot 2\pi/60 = 5\pi$ rad./sec.

Hence $v = 4 \cdot 5\pi = 20\pi$ ft./sec. $a_T = 0$ $a_N = 4(5\pi)^2 = 100\pi^2$ ft./sec^2.

(2) Calling the required acceleration k, we have $d\omega/dt = k$ $\therefore \omega = kt +$ some constant C (§ 104). But $\omega = 0$ when $t = 0$; hence $C = 0$, and $\omega = kt$. Therefore $5\pi = k\,60$, that is, $k = \pi/12$ rad./sec^2.

EXERCISE XIV

1. Prove by vector addition that the diagonals of a parallelogram bisect each other.

2. Show that when the speed of a motion is constant the hodograph is a circle or part of one.

3. A fly wheel of 5 ft. radius is making 240 revolutions per minute. For a point P on its rim find v and a in feet and seconds. If the center be at O, find v_x, v_y, a_x, a_y when $xOP = \pi/6$. How many feet will P travel in 3 minutes? What constant retardation (negative angular acceleration) would bring the wheel to rest in 3 minutes?

4. The equations of motion of P are $x = r\cos kt$, $y = r\sin kt$, k being positive. Show that P moves counterclockwise on the circle $x^2 + y^2 = r^2$ with the constant speed kr, and with the constant acceleration k^2r directed toward O.

5. The equations of motion of P are $x = 3\cos t$, $y = 2\sin t$. Show that P moves counterclockwise on the ellipse $x^2/9 + y^2/4 = 1$. Find v, a_T, a_N, a when $t = 0$; when $t = \pi/4$.

6. For each of the following motions find the path; also the vectors PV, PA when $t = 1$.

1. $x = t + 2, y = 1 - t/2$
2. $x = t + 1, y = t^2 + 2t$
3. $x = 2t, y = 3\sin t$.

7. A point is moving on the line $3x + 4y - 12 = 0$ in such a way that $x = 4t^2$. Find PV and PA when $t = 1/2$. Also find the hodograph.

8. Discuss the motion $x = 2\sin t$, $y = \cos 2t$, showing that it is oscillatory on the arc of the parabola $y = 1 - x^2/2$ between the points $(-2, -1)$ and $(2, -1)$, and that its hodograph is the curve $y^2 = x^2(4 - x^2)$.

9. A point P is moving away from O on the upper half of the curve $3y^2 = 2x^3$ with the constant speed 1 in./sec. Find PV and PA at $(6, 12)$.

10. A point P is moving on the hyperbola $xy = 4$ in such a manner that $v_x = 6$ in./sec. Express v_y and a_y in terms of x. Draw PV and PA when P is at $(2, 2)$.

11. A train is running toward a station at O on a track $y = x^2$ (1 m. being unit). If its speed at $(1, 1)$ is 50 m.h., how rapidly is it then approaching Ox, Oy, and O?

12. A point makes a circuit of the ellipse $x^2 - 4xy + 6y^2 = 34$. Prove that there are certain positions in which the y-component of its velocity is double the x-component, and find these positions.

13. A projectile P starts from O with the speed v_0 ft./sec. in a direction which makes the angle α with the horizontal Ox. The resistance of the air being neglected, the position of P at the time t (in ft., sec.) is

$$x = v_0 t \cos\alpha \qquad\qquad y = v_0 t \sin\alpha - \tfrac{1}{2} g t^2 \qquad (1)$$

where $g = 32$ is the acceleration of gravity. The equations (1) represent a parabola.

(1) Let R denote the *range*, that is, the distance from O to the point of Ox to which P falls; T, the time of flight; H, the greatest height. Prove that

$$R = \frac{v_0^2 \sin 2\alpha}{g} \qquad T = \frac{2 v_0 \sin\alpha}{g} \qquad H = \frac{v_0^2 \sin^2\alpha}{2g} \qquad (2)$$

(2) Show that R is greatest when $\alpha = \pi/4$, and $R = v_0^2/g$. Show also that there are two angles α which will give any range less than v_0^2/g.

(3) Find the speed v; also when and where it is least.

(4) What is the smallest v_0 that will give a range of ten miles?

14. A fielder is standing 300 ft. from the home plate when a ball is hit toward him for which $v_0 = 120$ ft./sec. and $\alpha = \pi/6$. How fast must he run to catch the ball?

15. A man can throw a ball with the speed 100 ft./sec. What elevation must he give the ball if he is to hit a mark whose coordinates, referred to his throwing hand as origin, are (100 ft., 50 ft.)?

XI. CURVE TRACING

89. Equations of the type $y^2 = f(x)$. The graph of an equation of this type is confined to portions of the plane in which $f(x)$ is positive or 0; for when $f(x)$ is negative, y is imaginary. To each positive value of $f(x)$ correspond the values $\pm\sqrt{f(x)}$ of y; hence the curve is symmetric with respect to Ox. In what follows, $f(x)$ is a rational function, integral or fractional.

EXAMPLE 1. Find the graph of $4\,y^2 = x^3 - 3\,x$ or $2\,y = \pm(x^3 - 3\,x)^{1/2}$

$$4\,f(x) = x^3 - 3\,x = (x + \sqrt{3})\,x\,(x - \sqrt{3})$$

Hence $f(x)$ is 0 at $x = -\sqrt{3}$, 0, $\sqrt{3}$, and is + for $-\sqrt{3} < x < 0$ and for $x > \sqrt{3}$, and then only. The corresponding parts of the curve are an oval and an infinite branch, both symmetric to Ox. For great values of x the dominant term of $x^3 - 3\,x$ is x^3; hence for such values of x the graph approximates the curve $4\,y^2 = x^3$ (in shape). Similarly for small values of x it approximates the curve $4\,y^2 = -3\,x$.

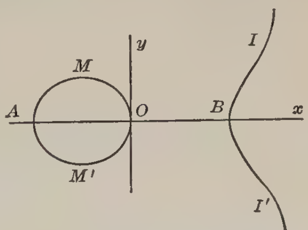

FIG. 63.

The turning points M, M' and the points of inflexion I, I' may be found as follows:

Differentiating the equation $4\,y^2 = x^3 - 3\,x$ (1) twice gives

$$8\,y\frac{dy}{dx} = 3\,x^2 - 3 \quad (2) \qquad\qquad 8\,y\frac{d^2y}{dx^2} + 8\left(\frac{dy}{dx}\right)^2 = 6\,x \quad (3)$$

By (2), $dy/dx = 0$ when[1] $3\,x^2 - 3 = 0$ $\therefore$ by (1) at the points $M(-1, 1/\sqrt{2})$, $M'(-1, -1/\sqrt{2})$

By (3), $d^2y/dx^2 = 0$ when[1] $4(dy/dx)^2 = 3\,x$ $\therefore$ by (2), (1), when $x^4 - 6\,x^2 - 3 = 0$ (4)

The equations (4) and (1) give the points $I(2.5, 1.4)$, $I'(2.5, -1.4)$.

[1] Since, by (1), $y \neq 0$ at these points.

EXAMPLE 2. Show that from the equation (2) in Ex. 1 it follows that the curve cuts Ox at right angles at the points A, O, and B.

90. Points on Ox. The curve $y^2 = f(x)$, call it C, meets Ox at points where $f(x)$ is 0. When $f(a)$ is 0, $f(x)$ can be expressed in the form $(x - a)^r\phi(x)$, where $r \gtreqless 1$ and $\phi(a) \neq 0$ or ∞, § 50, 2.

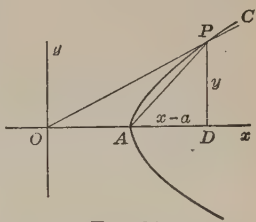

FIG. 64.

1. If, as is ordinarily the case, $r = 1$, the curve C cuts Ox at right angles at $A(a, 0)$. For let $P(x, y)$ be a point of C near A. The slope of AP is $y/(x - a)$; hence the slope of C at A is $\lim\limits_{P\to A} [y/(x - a)]$. But since $y^2 = (x - a)\phi(x)$, and $\phi(a) \neq 0$, we have

$$\lim_{P\to A}\left[\frac{y}{x-a}\right]^2 = \lim_{P\to A}\frac{\phi(x)}{x-a} = \infty$$

2. When $r > 1$, A is a *singular point*[1] of C. In particular, [see Fig. 65]

(1) When $r = 2$ so that $y^2 = (x - a)^2\phi(x)$, and $\phi(a)$ is $+$, then $\lim\limits_{P\to A} [y/(x - a)]^2 = \phi(a)$. Hence two branches of C pass through A with the slopes $\pm\sqrt{\phi(a)}$. We call A a *node*.

(2) When $r = 3$ or a greater odd number, A is the end point of two branches of C both of which touch Ox at A and extend to one side only of A. We call A a *cusp*.

(3) When $r = 4$ or a greater even number and $\phi(a)$ is $+$, two branches of C touch Ox at A and extend to both sides of A. We call A a *point of osculation*.

(4) When r is even and $\phi(a)$ is $-$, A is an isolated point of C. It is called a *conjugate point*.

[1] A point of any curve $F(x, y) = 0$ where both $\partial F/\partial x$ and $\partial F/\partial y$ are 0 is called a singular point (§ 42, 4.). For the case in which $F(x, y) \equiv y^2 - f(x)$, we have $\partial F/\partial x = -f'(x)$, $\partial F/\partial y = 2y$; and both $f'(x)$ and y are 0 at A when $r > 1$.

EXAMPLE 1. Find the graph of $y^2 = (x+1)^2x^3(x-1)^2(x-2)^4$, showing that

$A(-1, 0)$ is a *conjugate point*
$O(0, 0)$ is a *cusp*
$B(1, 0)$ is a *node*, slopes ± 2
$C(2, 0)$ is a *point of osculation*

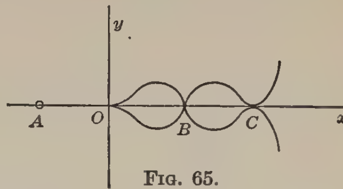

FIG. 65.

EXAMPLE 2. Find the graphs of the following equations:

1. $y^2 = x^2(a^2 - x^2)$ 2. $y^2 = ax^3 - x^4$ 3. $y^2 = x^4(a^2 - x^2)$
4. $y^2 = x^2(x^2 - a^2)$ 5. $y^2 = x^2(x - a)$ 6. $y^2 = x^3 + 1$

91. Infinite branches. Asymptotes. Call the curve C.

1. If $f(x) \to +\infty$ when $x \to b$ or when $x \to b$, then $x = b$ is an asymptote of a branch of C above Ox and of the corresponding branch below. Thus, $x = 0$ is an asymptote of $y^2 = x + 1/x$.

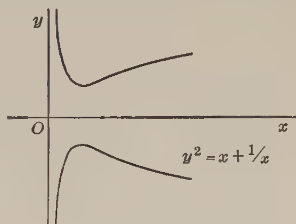

FIG. 66.

2. The slope of OP in Fig. 64 is y/x, and $(y/x)^2 = f(x)/x^2$. Let $x \to +\infty$; if $f(x)$ remains positive as x increases, the upper and lower halves of C extend infinitely far to the right, and according as $f(x)/x^2 \to \infty$, 0, or $l(>0)$, their directions will approach those of Oy, Ox, or the lines $y = \pm l^{1/2}x$. Similarly when $x \to -\infty$.

3. *The curve C has oblique or horizontal asymptotes if $f(x)$ can be reduced to the form $u^2 + v$, where u is real and of the first degree in x or constant, and $v/u \to 0$ when $x \to \infty$, the asymptotes being $y = u$ and $y = -u$.*

For
$$\pm y = (u^2+v)^{1/2} = |u| + [(u^2+v)^{1/2} - |u|]$$
$$= |u| + \frac{v}{[(u^2+v)^{1/2} + |u|]}$$
and the fractional term $\to 0$ when $v/u \to 0$. $\therefore$ when $x \to \infty$. Also $\lim\limits_{x\to\infty} y/x = \pm \lim\limits_{x\to\infty} u/x =$ the slope of $y = \pm u$. Hence the lines $y = u$ and $y = -u$ are asymptotes, § 51, 2.

4. This reasoning also shows that the graph of

$$(y - mx - c)^2 = u^2 + v \quad \text{or} \quad y = mx + c \pm (u^2 + v)^{1/2}$$

has the asymptotes $y = mx + c + u$ and $y = mx + c - u$.

EXAMPLE 1. Find the asymptotes of the hyperbola $y^2 = x^2 + 2x$.

$$x^2 + 2x = (x+1)^2 - 1, \text{ and } \lim_{x\to\infty} [1/(x+1)] = 0$$

Hence the asymptotes are $y = x + 1$, $y = -(x + 1)$.

EXAMPLE 2. Solving $y^2 - 2xy + 1 = 0$ for y gives

$$y = x \pm (x^2 - 1)^{1/2}; \text{ and } \lim_{x\to\infty} \frac{1}{x} = 0$$

Hence the graph (an hyperbola) has the asymptotes $y = x \pm x$, or $y = 2x$ and $y = 0$.

EXAMPLE 3. Show that $y^2 = (x + 1)/x = 1 + 1/x$ has the asymptotes $y = 1$, $y = -1$ and $x = 0$.

EXAMPLE 4. Trace the curve $y^2 = (x^3 + x^2)/(x - 1)$. The points on Ox, given by $(x^3 + x^2) = (x + 1)x^2 = 0$, are $A(-1, 0)$ and O; and O is a conjugate point since $f(x)$ is $+$ only when $x < -1$ or $x > 1$. The line $x - 1 = 0$ is an asymptote. Also, dividing by $x - 1$,

$$f(x) = x^2 + 2x + 2 + \frac{2}{x-1}$$
$$= (x+1)^2 + \left(1 + \frac{2}{x-1}\right),$$

and

$$\left(1 + \frac{2}{x-1}\right)/(x+1) \to 0 \qquad \text{when } x \to \infty.$$

Hence the lines $y = x + 1$ and $y = -(x + 1)$ are asymptotes.

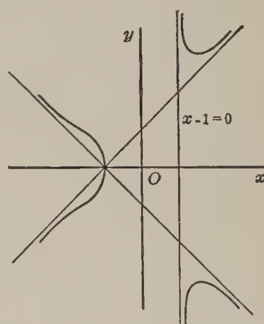

FIG. 67.

EXAMPLE 5. Show that the curve $y^2 = x^3(x - 1)^2/(x + 1)^3$ has a cusp at O, a node at $(1, 0)$, and the asymptotes $x + 1 = 0$, $y = x - 5/2$, $y = -x + 5/2$. Trace it.

EXAMPLE 6. Trace the following curves:

1. $y^2 = (x-1)/x$	2. $y^2 = (1-x)/x$	3. $y^2 = (1-x)^3/x$
4. $y^2 = (x-1)^3/x$	5. $y^2 = x^2/(a^2 - x^2)$	6. $y^2 = x^2/(x^2 - a^2)$
7. $y^2 = (x^2+x^3)/(1-x)$	8. $y^2 = x^2/(x^2+1)$	9. $y^2 = x^4/(x^2+1)$

92. Conics. The general equation of the second degree in x, y is

$$ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c = 0$$

where a, h, $\cdots$ denote constants. The examples $x^2 - y^2 = 0$, $x^2 + y^2 = 0$, $x^2 + y^2 + 1 = 0$ show that such an equation may represent a pair of straight lines, or a conjugate point, or may have no real solution and therefore no graph. In all other cases it represents a conic, and this conic is an

ellipse, parabola, or hyperbola according as $h^2 - ab \lesseqgtr 0$

EXAMPLE 1. Trace the curve $x^2 - 2\,xy + 2\,y^2 - 2\,x = 0$.

Solving for y, $$y = \frac{x}{2} \pm \frac{(4\,x - x^2)^{1/2}}{2}$$

$4\,x - x^2 = x(4 - x)$ is 0 at $x = 0$ and 4, and + when $0 < x < 4$

Hence the curve lies between the lines $x = 0$ and $x = 4$. Draw these lines and also the "guide line" $y = x/2$. Then for any $x = OX$ between $x = 0$ and $x = 4$ we get two curve points P_1 and P_2 by decreasing and increasing the ordinate XQ of $y = x/2$ by the value of $(4\,x - x^2)^{1/2}/2$ for $x = OX$. When X is at O or A, the points P_1, P_2 coincide; the curve touches Oy at O and AB at B. Since the curve is a conic and is closed it is an *ellipse*; observe that $h^2 - ab = 1^2 - 1 \cdot 2 = - 1 < 0$.

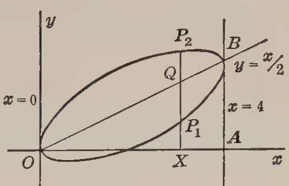

FIG. 68.

Since OB bisects a system of parallel chords, it is a *diameter*. Its mid-point is the *center* of the ellipse. The highest and lowest curve points may be found by solving the given equation $f(x, y) = 0$ and $\partial f/\partial x = 2\,x - 2\,y - 2 = 0$ for x, y.

EXAMPLE 2. Trace the curve $(y - x)^2 = x + 2$.

Solving for y, $y = x \pm (x + 2)^{1/2}$.

The curve extends indefinitely to the right of the line $x + 2 = 0$ and tends to parallelism with the guide line $y = x$. As the curve is a conic and consists of a single infinite part it is a *parabola*. It cuts Ox at $x = - 1$ and 2; and Oy at $y = \pm \sqrt{2}$.

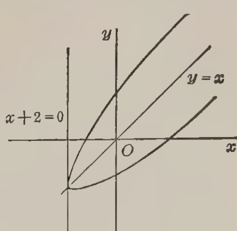

FIG. 69.

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EXAMPLE 3. Trace the curve $y^2 + xy - 2x^2 - 4 = 0$.

Solving for y, $y = -\frac{x}{2} \pm \frac{(9x^2+16)^{1/2}}{2}$.

The radicand is of the form $u^2 + v$ where $\frac{v}{u} = \frac{16}{3x} \to 0$ when $x \to \infty$. Hence, § 91, 3., the curve has the asymptotes $y = -\frac{x}{2} \pm \frac{3x}{2}$, that is, $y = x$ and $y = -2x$; and as $(9x^2+16)^{1/2} > 3|x|$, it lies with respect to these asymptotes as in Fig. 70. Since it is a conic with asymptotes, it is an *hyperbola*.

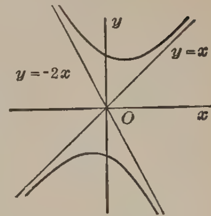

FIG. 70.

EXAMPLE 4. Trace the following:

1. $y^2 - x^2 - 2y = 0$
2. $x^2 - xy + y^2 - 3 = 0$
3. $4y^2 + 4xy + x^2 + x = 0$
4. $y^2 + xy - 2x^2 + 4 = 0$
5. $y^2 - 2xy + 2x = 0$
6. $y^2 + 4x^2 - 2x + 2y = 0$
7. $y^2 - 2xy + 2x^2 - x - 2 = 0$
8. $y^2 - 2xy - 3x^2 + 2x - 2y + 1 = 0$

93. Other algebraic curves. The graphs of algebraic equations $f(x, y) = 0$ of a degree higher than two in both y and x will be considered later. The following example illustrates a method applicable to equations of degree $n > 1$ in x, y, which contain terms of degrees n and $n-1$ only.

EXAMPLE 1. Trace the curve $x^3 + y^3 - 3axy = 0$ (1) [*Folium of Descartes*]

The equation $y = tx$ (2) represents the line OP through O with the slope t.

Solving (1), (2) for x, y shows that OP cuts the curve at O twice and at the point

$$P: x = \frac{3at}{t^3+1} \quad y = \frac{3at^2}{t^3+1} \quad (3)$$

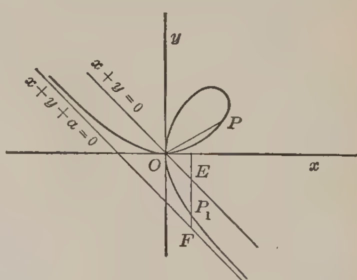

FIG. 71.

From (3) it follows that when $t \to 0$, then $P \to O$, and $OP \to Ox$; also when $t \to \infty$, then $P \to O$ and $OP \to Oy$. Hence Ox and Oy are tangents to the curve at O.

When $t \to -1$, x and y become infinite. This leads us to look for an asymptote parallel to the line $y + x = 0$. We find this asymptote to be the line $y + x + a = 0$ (4). For [footnote, p. 53]

$$EP_1 = y + x = \frac{3\,at^2 + 3\,at}{t^3 + 1} = \frac{3\,at}{t^2 - t + 1}, \text{ which } \to -a \text{ when } t \to -1.$$

Hence $FP_1 \to 0$ when $t \to -1$: that is, $y + x + a = 0$ is an asymptote. Trace the curve as t increases from $-\infty$ to ∞.

EXAMPLE 2. Show that the curve $y^3 - x^3 = 3\,x^2$ has the asymptote $y = x + 1$ and a cusp at O with the tangent $x = 0$, and also meets Ox at $x = -3$. Trace the curve.

94. The cycloid. Suppose a circle to roll, without sliding, on a straight line; the curve traced by any point on its cir-

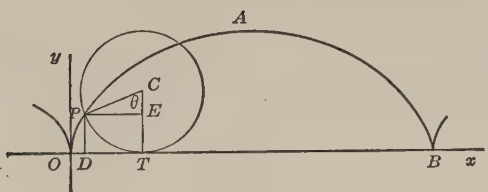

FIG. 72.

cumference is called a *cycloid*. In Fig. 72 the line is Ox, the circle rolls to the right and the initial position of $P(x, y)$ is O.

Let a = radius, $\theta = TCP$. Then

$$OT = \text{arc } TP = a\theta,$$
$$x = OD = OT - PE = a\theta - a\sin\theta$$
$$y = DP = TC - EC = a - a\cos\theta.$$

Hence the equations of this cycloid are

$$x = a(\theta - \sin\theta) \qquad y = a(1 - \cos\theta) \qquad (1)$$

As θ increases clockwise from 0 to $2\,\pi$, P traces the arch OAB; O and B are cusps.

95. Epicycloid, hypocycloid. 1. As the circle CP rolls on the circle OA, Fig. 73, the point $P(x, y)$, starting at A, traces an *epicycloid*.

Let $a = OT$, $b = TC$, $\theta = AOT$, $\phi = TCP$. Then

$$x = OD = OF + EP = (a+b)\cos\theta + b\sin ECP$$
$$y = DP = FC - EC = (a+b)\sin\theta - b\cos ECP$$

But arc AT = arc TP $\therefore a\theta = b\phi$;

also $ECP = \phi - \left(\frac{\pi}{2} - \theta\right)$.

Hence

$$x = (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta,$$

$$y = (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta \qquad (1)$$

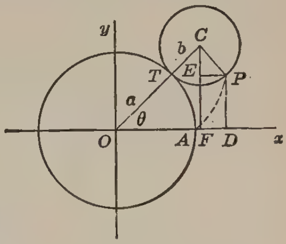

FIG. 73.

2. When the rolling circle is inside, the point P traces a *hypocycloid* whose equations — got by replacing b by $-b$ in the equations (1) — are

$$x = (a-b)\cos\theta + b\cos\frac{a-b}{b}\theta, \quad y = (a-b)\sin\theta - b\sin\frac{a-b}{b}\theta \quad (2)$$

EXERCISE XV

1. Show that the hypocycloid for which $a = 2b$ is the horizontal diameter of the fixed circle counted twice. Prove this also by elementary geometry.

2. Show that when $a = 4b$, the equations of § 95 (2) reduce to $x = a\cos^3\theta$, $y = a\sin^3\theta$ $\therefore$ to $x^{2/3} + y^{2/3} = a^{2/3}$. The curve is the *hypocycloid of four cusps.* Trace it.

3. Show that the epicycloid for which $b = a$ is a cardioid (p. 78, Fig. 45) whose polar equation referred to the point A in Fig. 73 as origin is $r = 2a(1 - \cos\theta)$.

4. Show that the slope of the cycloid at the point P, Fig. 72, is $\cot\theta/2$; hence that the tangent at P passes through the highest point F of the rolling circle, and that PT is the normal[1] at P and has the length $2a\sin\theta/2$.

[1] This may also be shown as follows: When the circle, starting from the position in Fig. 72, turns through the angle $\delta\theta$, the point T of the circle moves a distance ϵ which is very small as compared with $\delta\theta$ and such that $\epsilon/\delta\theta \to 0$ when $\delta\theta \to 0$. Hence we may regard the motion of the circle as at every instant one of rotation about its point of tangency T, which means that at the instant P is moving in a direction perpendicular to TP. Similarly, in Fig. 73, TP is normal to the curve that is being traced by P.

5. Prove that the radius of curvature of the cycloid at P is $4a \sin \theta/2 = 2\,PT$. Hence show geometrically that the evolute of the cycloid OAB is an equal cycloid generated by a point Q of the circumference of a circle of radius a which rolls on the line $y = -2a$.

6. Show that a point Q on the radius CP of the rolling circle in Fig. 72, such that $CQ = b$, generates the curve $x = a\theta - b \sin \theta$, $y = a - b \cos \theta$ (a *trochoid*).

7. How many cusps has an epicycloid for which $b/a = 2/3$? $\sqrt{2}$?

8. Show that when $d\theta/dt$ is constant the speed of P in Fig. 72 is proportional to PT.

9. A rolling wheel of 2 ft. radius moves along the track at the rate of 60 ft./sec. What is its angular velocity $d\theta/dt$? Find, in magnitude and direction, the velocity and acceleration of a point P on the rim when $\theta = 0, \pi/2, 3\pi/4, \pi, 3\pi/2$.

10. Show that $8y^2 + 12xy + 17x^2 = 20$ (1) represents an ellipse with its center at O. Show that its axes can be determined by finding the maximum and minimum values of $u = x^2 + y^2$ (2) subject to the condition (1) by the method of § 49, and that the major axis is on the line $2y - x = 0$, the minor axis on the line $y + 2x = 0$, their lengths being 4 and 2. Trace the curve.

11. Trace the curves:

1. $x^2 = y(y-1)^2(y-2)^3$
2. $y^2 - 2xy + x^4 = 0$
3. $y^3 - x^2y - x^4 = 0$
4. $x^2y - 2xy^2 + x^2 - y^2 = 0$

XII. MEAN VALUE THEOREM. INDETERMINATE FORMS

96. Rolle's Theorem. *Let $f(x)$ be a function which vanishes at $x = a$ and at $x = b$, and has a finite derivative at all points in (a, b). Then $f'(x)$ vanishes at some point x_1 between (and distinct from) a and b.*

The geometrical meaning of the theorem is that between the points $x = a$ and $x = b$ where the curve $y = f(x)$ meets Ox it has at least one turning point. Its arithmetical proof is as follows:

Since $f(x)$ is continuous in (a, b), § 25, 2., it takes least and greatest values, m and M, in (a, b), § 18, 1. And unless $f(x)$ is 0 throughout (a, b) — in which case $f'(x)$ also is 0 throughout (a, b) and the theorem is true — at least one of m and M is different from 0 and therefore corresponds to a value of x distinct from a and b, where $f(x)$ is 0. Let this value of x be x_1. Then $f'(x_1)$ is 0. For $f'(x_1)$ is finite by hypothesis; and were it positive or negative, $f(x_1)$ could not be m or M, since in (a, b), near x_1, there would be values of $f(x)$ less, and values greater, than $f(x_1)$, § 31.

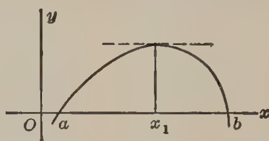

FIG. 74.

97. Mean value theorem. *Let $f(x)$ be a function which has a finite derivative at all points of the interval (a, b). There exists a value x_1 of x between (and distinct from) a and b such that*

$$f(b) - f(a) = f'(x_1)(b - a) \tag{1}$$

For there is a number k such that $f(b) - f(a) = k(b - a)$, or that

$$f(b) - f(a) - k(b - a) = 0 \tag{2}$$

If we replace b by x in the first member of (2), we obtain a function

$$F(x) = f(x) - f(a) - k(x - a) \tag{3}$$

which satisfies the conditions of Rolle's theorem, § 96. For $F(a)$ is 0 identically, $F(b)$ is 0 by (2), and $F'(x) = f'(x) - k$ is finite throughout (ab). Hence there exists a value x_1 of x between, and distinct from, a and b such that

$$F'(x_1) = f'(x_1) - k = 0 \ \therefore \text{ that } f'(x_1) = k \tag{4}$$

But the substitution of $f'(x_1)$ for k in (2) gives (1); as was to be proved.

Any number between a and b may be expressed in the form $a + \theta(b - a)$ where θ denotes some number between 0 and 1. Hence (1) may be written

$$f(b) - f(a) = f'[a + \theta(b - a)](b - a) \tag{5}$$

If in (5) we replace a by x and b by $x + \Delta x$, $b - a$ becomes Δx and $f(b) - f(a)$ becomes $f(x + \Delta x) - f(x)$, or Δy if $y = f(x)$. We therefore have

$$\Delta y = f'(x + \theta\Delta x)\Delta x \tag{6}$$

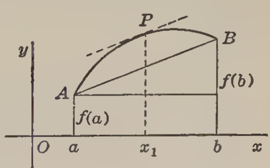

FIG. 75.

Respecting the graph of $y = f(x)$, the mean value theorem asserts that on the arc between the points $A[a, f(a)]$ and $B[b, f(b)]$ there is a point $P[x_1, f(x_1)]$ where the tangent is parallel to the chord AB; for by (1)

$$\frac{f(b) - f(a)}{b - a} = f'(x_1) \tag{7}$$

The first member of (7) is the slope of AB; the second that of the tangent at P.

EXAMPLE 1. Find x_1 such that $f(b) - f(a) = f'(x_1)(b - a)$ in the following cases:

1. $f(x) = x^2$, $a = 2$, $b = 5$
2. $f(x) = x^3$, $a = 1$, $b = 2$
3. $f(x) = x^{1/2}$, $a = 1$, $b = 4$

Example 2. Given a curve C: $y = f(x)$, and three points P_1, P_2, P_3 on C. Ordinarily when P_2, $P_3 \to P_1$ on C, the circle determined by P_1, P_2, P_3 approaches a definite circle S as limit. This circle S is called the *osculating circle* to C at P_1. We are to prove that it is the circle of curvature of C at P_1. We suppose $f''(x)$ continuous between P_1, P_2, P_3.

Represent the equation of the circle $P_1P_2P_3$ by

$$(x - a)^2 + (y - b)^2 - r^2 = 0 \tag{1}$$

and consider the function $$F(x) = [x - a]^2 + [f(x) - b]^2 - r^2 \tag{2}$$

Since $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ are on the circle (1), $F(x)$ vanishes at $x = x_1, x_2, x_3$.

Hence, by Rolle's theorem, § 96, $F'(x)$ vanishes at a point x_1' between x_1 and x_2 and also at a point x_2' between x_2 and x_3. And since $F'(x)$ vanishes at x_1' and x_2', $F''(x)$ vanishes at a point x_1'' between x_1' and x_2'. When $x_2, x_3 \to x_1$, then $x_1', x_2', x_1'' \to x_1$.

Hence when $x_2, x_3 \to x_1$ and (1) becomes the equation of the osculating circle at P_1, the constants a, b, r^2 satisfy the equations $F(x_1) = 0$, $F'(x_1) = 0$, $F''(x_1) = 0$, that is, the equations:

$$(x_1 - a)^2 + [f(x_1) - b]^2 - r^2 = 0, \qquad (x_1 - a) + [f(x_1) - b]\, f'(x_1) = 0,$$
$$1 + f'(x_1)^2 + [f(x_1) - b]\, f''(x_1) = 0$$

But these equations when solved for a, b, and r give the values found in § 80, (4) and § 79, (2) for the center (α, β) and the radius R of the circle of curvature.

Hence the circle of curvature at P_1 crosses C at P_1, as in Fig. 53.

98. Corollary. *If $f'(x)$ is 0 throughout (a, b), then $f(x)$ is constant throughout (a, b).*

For if x denote any number in (a, b), we have, by § 97, (1),

$$f(x) - f(a) = f'(x_1)(x - a) \quad \text{where} \quad a < x_1 < x$$

But $f'(x_1) = 0 \therefore f(x) - f(a) = 0 \therefore f(x)$ has the constant value $f(a)$ in (a, b).

INDETERMINATE FORMS

99. The form 0/0. If $F(x)$ takes an indeterminate form when $x = a$, but $\lim\limits_{x \to a} F(x)$ exists, it is customary to assign to $F(a)$ the value $\lim\limits_{x \to a} F(x)$, § 13.

1. If the fraction $f(x)/\phi(x)$ takes the form $0/0$ when $x = a$, it is often possible, as in § 13, to discover a factor common to

$f(x)$ and $\phi(x)$ which vanishes when $x = a$, and after the removal of this factor to find $\lim_{x \to a} f(x)/\phi(x)$ directly.

EXAMPLE. When $x = 0$, $\sin 2x/\tan x$ becomes $0/0$; find

$$\lim_{x \to 0} \sin 2x/\tan x$$

$$\frac{\sin 2x}{\tan x} = \frac{2 \sin x \cos x}{\sin x/\cos x} = 2 \cos^2 x; \text{ hence } \lim_{x \to 0} \frac{\sin 2x}{\tan x} = \lim_{x \to 0} 2 \cos^2 x = 2.$$

2. A more general method is supplied by the following theorems:

If $f(x)$ and $\phi(x)$ have finite derivatives throughout the interval (a, b), and if $\phi'(x)$ does not vanish in (a, b) except perhaps at a or b, then there exists in (a, b) a value x_1 of x, distinct from a and b, such that

$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(x_1)}{\phi'(x_1)}. \tag{1}$$

For let k be the value of $[f(b) - f(a)]/[\phi(b) - \phi(a)]$, so that

$$f(b) - f(a) - k[\phi(b) - \phi(a)] = 0 \tag{2}$$

If we replace b by x in the first member of (2), we obtain a function

$$F(x) = f(x) - f(a) - k[\phi(x) - \phi(a)] \tag{3}$$

which satisfies the conditions of Rolle's theorem; for $F(a) = 0$ identically, $F(b) = 0$ by (2), and $F'(x) = f'(x) - k\phi'(x)$ is finite throughout (a, b). Hence, by § 96, there exists a value x_1 of x, between and distinct from a and b, such that $F'(x_1) = f'(x_1) - k\phi'(x_1) = 0$, therefore, since $\phi'(x_1) \neq 0$, such that $f'(x_1)/\phi'(x_1) = k$; which proves (1).

If $f(a) = 0$ and $\phi(a) = 0$, and $\lim_{x \to a} \frac{f'(x)}{\phi'(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}$$

For in (1) replace b by x. Then since $f(a)=0$, $\phi(a)=0$, (1) becomes $\dfrac{f(x)}{\phi(x)}=\dfrac{f'(x_1)}{\phi'(x_1)}$, where x_1 is between a and x. When $x \to a$, then $x_1 \to a$. Hence $\lim\limits_{x\to a}\dfrac{f(x)}{\phi(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{\phi'(x)}$.

Therefore $\lim\limits_{x\to a} f(x)/\phi(x)=f'(a)/\phi'(a)$ or ∞, unless both $f'(a)$ and $\phi'(a)$ are 0.

If $f'(a)=0$ and $\phi'(a)=0$ but $\lim\limits_{x\to a} f''(x)/\phi''(x)$ exists, then $\lim\limits_{x\to a} f'(x)/\phi'(x)=\lim\limits_{x\to a} f''(x)/\phi''(x)$, from which it follows that $\lim\limits_{x\to a} f(x)/\phi(x)=f''(a)/\phi''(a)$ or ∞, unless both $f''(a)$ and $\phi''(a)$ are 0. And so on.

Before each step in this process the fraction under consideration should be reduced to its simplest form. Sometimes a transformation is necessary in order to avoid a never ending sequence of indeterminate forms.

Example 1. At $x=1$, $(\log x)/(x^2-x)$ becomes $0/0$; find its limiting value when $x \to 1$.

$$\frac{f'(x)}{\phi'(x)}=\frac{1/x}{2x-1} \qquad \therefore \frac{f'(1)}{\phi'(1)}=\frac{1}{2-1}=1, \text{ the limiting value required.}$$

Example 2. At $x=0$, $(e^x+e^{-x}-2)/x^2$ becomes $0/0$; find its limiting value when $x \to 0$.

$$\frac{f'(x)}{\phi'(x)}=\frac{e^x-e^{-x}}{2x} \qquad \therefore \frac{f'(0)}{\phi'(0)}=\frac{0}{0}$$

$$\frac{f''(x)}{\phi''(x)}=\frac{e^x+e^{-x}}{2} \qquad \therefore \frac{f''(0)}{\phi''(0)}=1, \text{ the limiting value required.}$$

Example 3. Show that the following are indeterminate and find their limiting values.

1. $\dfrac{\log x}{x-1}$ at $x=1$

2. $\dfrac{\tan 2\theta-2\tan\theta}{2\sin\theta-\sin 2\theta}$ at $\theta=0$

3. $\dfrac{e^x-e^{-x}-2\sin x}{4x^3}$ at $x=0$

100. The form ∞/∞. It can also be proved[1] that *when $f(a)/\phi(a)$ has the form ∞/∞, but $\lim\limits_{x \to a} f'(x)/\phi'(x)$ exists, then $\lim\limits_{x \to a} f(x)/\phi(x) = \lim\limits_{x \to a} f'(x)/\phi'(x)$.*

EXAMPLE. When $x = 0$, $\log x/\cot x$ becomes ∞/∞; find its limiting value when $x \to 0$.

$$\frac{f'(x)}{\phi'(x)} = \frac{1/x}{-\operatorname{cosec}^2 x} = -\frac{\sin^2 x}{x} \quad \therefore \frac{f'(0)}{\phi'(0)} = \frac{0}{0}; \quad \frac{f''(x)}{\phi''(x)} = -\frac{2\sin x \cos x}{1}$$

$$\therefore \frac{f''(0)}{\phi''(0)} = 0, \text{ the limiting value required.}$$

101. The forms $0 \cdot \infty$ and $\infty - \infty$. The form $0 \cdot \infty$ can be reduced to either of the forms $0/0$ or ∞/∞. The same is often true of the form $\infty - \infty$.

EXAMPLE 1. At $x = \pi/2$, $(1 - \sin x)\tan x$ becomes $0 \cdot \infty$; find its limiting value when $x \to \pi/2$.

$$(1 - \sin x)\tan x = \frac{1 - \sin x}{\cot x} = \frac{0}{0} \text{ when } x = \frac{\pi}{2};$$

but
$$\lim_{x \to \pi/2} \frac{1 - \sin x}{\cot x} = \lim_{x \to \pi/2} \frac{\cos x}{\operatorname{cosec}^2 x} = \frac{0}{1} = 0.$$

[1] 1. *When a is ∞.* Take an arbitrary constant c and suppose $x > c$. In (c, x) there is an x_1 such that (§ 99 (1))

$$\frac{f(x) - f(c)}{\phi(x) - \phi(c)} = \frac{f'(x_1)}{\phi'(x_1)}, \text{ and therefore } \frac{f(x)}{\phi(x)} = \frac{f'(x_1)}{\phi'(x_1)} \cdot \frac{1 - \phi(c)/\phi(x)}{1 - f(c)/f(x)}$$

Let l denote $\lim\limits_{x \to \infty} \dfrac{f'(x)}{\phi'(x)}$, and ϵ, ϵ' positive numbers as small as we please. We can take c large enough to make $\left|\dfrac{f'(x_1)}{\phi'(x_1)} - l\right| < \epsilon$, and then, keeping c fixed, we can take x large enough to make $\left|\dfrac{1 - \phi(c)/\phi(x)}{1 - f(c)/f(x)} - 1\right| < \epsilon'$; hence $\lim\limits_{x \to \infty} f(x)/\phi(x) = \lim\limits_{x \to \infty} f'(x)/\phi'(x) = l$.

If $\lim\limits_{x \to \infty} f'(x)/\phi'(x) = \infty$, then $\lim\limits_{x \to \infty} f(x)/\phi(x) = \infty$.

2. *When a is finite.* Set $x = a + 1/z$, so that when $x \to a$ then $z \to \infty$. Then if $f(x)$, $\phi(x)$ become $F(z)$, $\Phi(z)$, we have $f'(x) = -F'(z)z^2$, and $\phi'(x) = -\Phi'(z)z^2 \quad \therefore f'(x)/\phi'(x) = F'(z)/\Phi'(z)$.

Let $\lim\limits_{x \to a} f'(x)/\phi'(x) = l$. Then $\lim\limits_{z \to \infty} F'(z)/\Phi'(z) = l$ and therefore, by 1., $\lim\limits_{z \to \infty} F(z)/\Phi(z) = l$. Hence $\lim\limits_{x \to a} f(x)/\phi(x) = l$.

EXAMPLE 2. At $x = \pi/2$, $\sec x - \tan x$ becomes $\infty - \infty$; find its limiting value when $x \rightarrow \pi/2$.

$$\sec x - \tan x = \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1 - \sin x}{\cos x} = \frac{0}{0} \text{ when } x = \frac{\pi}{2};$$

but
$$\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x \to \pi/2} \frac{\cos x}{\sin x} = 0.$$

102. The forms 1^∞, 0^0, ∞^0. A function of the type $f(x)^{\phi(x)}$ may take one of the indeterminate forms 1^∞, 0^0, ∞^0. For if $u = f(x)^{\phi(x)}$, then $\log u = \phi(x) \log f(x)$, and at the point $x = a$ this product may take the form $0 \cdot \infty$ in the following cases:

(1) When $\phi(a) = \infty$ and $f(a) = 1$, u then having the form 1^∞.

(2) When $\phi(a) = 0$ and $f(a) = 0$ or ∞, u then having the form 0^0 or ∞^0.

In all these cases we find $\lim\limits_{x \to a} \log u$ by the method of § 101. Then if b denote this limit, we have, by § 67, 2, $\lim\limits_{x \to a} u = e^b$.

EXAMPLE. If $u = (1 + a/x)^x$, find $\lim\limits_{x \to \infty} u$ [the case 1^∞].

$$\log u = x \log(1 + a/x) = \frac{\log(1 + a/x)}{1/x}, \text{ which becomes } \frac{0}{0} \text{ when } x = \infty.$$

Hence $$\lim_{x \to \infty} \frac{\log(1 + a/x)}{1/x} = \lim_{x \to \infty} \frac{a/x^2}{(1 + a/x)(1/x^2)} = \lim_{x \to \infty} \frac{a}{1 + a/x} = a.$$

Therefore $\lim\limits_{x \to \infty} u = e^a$.

EXERCISE XVI

Find the following limiting values 1–21:

1. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{\sin 2x}$

2. $\lim\limits_{x \to a} \dfrac{\sin x - \sin a}{x - a}$

3. $\lim\limits_{x \to \pi/2} \dfrac{\sec x + 1}{\tan x}$

4. $\lim\limits_{x \to 0} x \log x$

5. $\lim\limits_{x \to \infty} \dfrac{x^n}{e^x}$

6. $\lim\limits_{x \to \infty} [\log(1 + x) - \log x]$

7. $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x - \sin x}$

8. $\lim\limits_{x \to 0} \dfrac{\sin x + \tan x}{e^x + e^{-x} - 2}$

9. $\lim_{x \to 0} \frac{xe^{4x} - x}{1 - \cos 2x}$

10. $\lim_{x \to 5} \frac{2 - \sqrt{x - 1}}{x^2 - 25}$

11. $\lim_{x \to 0} \frac{e^x - e^{-x} - \sin 2x}{x^3 + x^4}$

12. $\lim_{x \to 1} \frac{\sin \pi x}{x - 1}$

13. $\lim_{x \to 0} \frac{\log \sin x}{\log \tan x}$

14. $\lim_{x \to 0} \frac{\tan x}{\tan 4x}$

15. $\lim_{x \to 0} \sin x \log 2x$

16. $\lim_{x \to 1/2} \frac{\log (1 - 2x)}{\tan \pi x}$

17. $\lim_{x \to 0} \left[\frac{1}{\sin x} - \frac{1}{x}\right]$

18. $\lim_{x \to 0} \left[\frac{2}{\sin^2 x} - \frac{1}{1 - \cos x}\right]$

19. $\lim_{x \to 0} x^x$

20. $\lim_{x \to \infty} (1 + x)^{1/x}$

21. $\lim_{x \to 0} (e^{2x} + x)^{1/x}$

22. By aid of the mean value theorem, prove that if $f'(x)$ is positive throughout (a, b) and x', x'' are any two numbers in (a, b) such that $x' < x''$, then $f(x') < f(x'')$. Compare § 32.

23. Prove that if $f'(x)$ is continuous in (a, b) and M denote the greatest value of $|f'(x)|$ in (a, b), then in every part of (a, b) of length $< \epsilon/M$, the difference between the greatest and least values of $f(x)$ is less than ϵ.

XIII. NEWTON'S METHOD OF APPROXIMATION

103. Newton's method of approximation. An equation $f(x) = 0$ being given, suppose that two numbers b and c have been found such that

(1) $f(b)$ and $f(c)$ have opposite signs,

(2) $f'(x)$ and $f''(x)$ are continuous and $\neq 0$ in (b, c).

The equation $f(x) = 0$ then has one and but one root β in (b, c), §§ 18, 32, and we can find as close an approximation to β as we please by the following method, due to Newton.

Since by (2) the signs of $f'(x)$ and $f''(x)$ are constant in (b, c), it follows from §§ 32, 46 that the graph of $y = f(x)$ in (b, c) must be of the type of the curve arc in Fig. 76, 1. or 2., or of an arc symmetric to 1. or 2. with respect to Ox.

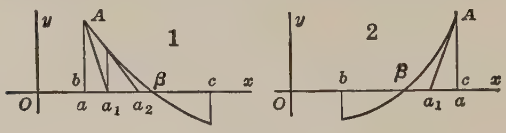

Fig. 76.

Hence if a denote that one of b or c for which $f(x)$ *has the sign of* $f''(x)$, the tangent at the corresponding curve point A will cut Ox at a point a_1 *between* a and β — which means that a_1 is a closer approximation to β than a is.

The equation of the tangent at A is $y - f(a) = f'(a)(x - a)$. Hence we can find a_1 by setting $y = 0$ in this equation and then solving for x. We thus get

$$a_1 = a - \frac{f(a)}{f'(a)} \tag{1}$$

Next, using a_1 as we have just used a, we get a second and closer approximation to β, namely $a_2 = a_1 - f(a_1)/f'(a_1)$; and so on.

As n increases, either $a_n \nearrow \beta$ as in Fig. 76, 1., or $a_n \searrow \beta$ as in Fig. 76, 2.[1]

Example 1. Find the real roots of $e^x + x - 2 = 0$ $(e = 2.71828 \cdots)$.

$$f(x) = e^x + x - 2 \qquad f'(x) = e^x + 1 \qquad f''(x) = e^x$$

Since $f(0) = -1$ and $f(1) = 1.718$ have opposite signs, and $f'(x)$ is always $+$, there is one and but one real root β, and it is in the interval $(0, 1)$. And since $f''(x)$ is always $+$ and therefore $\operatorname{sgn} f(1) = \operatorname{sgn} f''(1)$, we may begin the reckoning for β by setting $a = 1$ in the formula (1). But the reckoning will be shortened if we start with a smaller value of a. We find $f(.5) = .1487$, which is also $+$, and therefore take $a = .5$. The 1st and 2nd approximations to β are then

$$a_1 = .5 - \frac{f(.5)}{f'(.5)} = .4439 \qquad a_2 = .4439 - \frac{f(.4439)}{f'(.4439)} = .442884.$$

Example 2. Solve the equations (1) $2 \sin x = x$ (2) $e^x - 3x = 0$.

Example 3. Show that $x^3 - 2x - 5 = 0$ has the root 2.09455 [Newton's example].

Example 4. From the graphs of $y = \tan x$ and $y = x$ show that the equation $\tan x = x$ has infinitely many roots; and find the smallest positive root to the fourth decimal figure.

[1] The equation $f(x) = 0$ may be written in the form $x = \phi(x)$, where $\phi(x) = x - f(x)/f'(x)$. By (1) we then have $a_1 = \phi(a)$ $\therefore$ $a_2 = \phi(a_1), \cdots$, $a_n = \phi(a_{n-1})$ As n increases, a_n steadily increases or decreases but remains finite and therefore approaches a limit l, § 5. Therefore, since $\phi(x)$ is continuous in (b, c), $l = \phi(l)$, that is, $l = \beta$.

XIV. INTEGRATION

104. Integrals. Let $f(x)$ denote a given function of x; a function $F(x)$ which has $f(x)$ for its derivative, or $f(x)dx$ for its differential, is called an *integral* of $f(x)$ or of $f(x)dx$. Thus x^2 is an integral of $2x$ or of $2xdx$.

The derivative of any constant C being 0, if $F(x)$ is an integral of $f(x)$, so also is $F(x)+C$. Furthermore

If $f(x)$ is continuous and has the integral $F(x)$ in the interval (a, b), then every integral of $f(x)$ in (a, b) is of the form $F(x)+C$.

For let $G(x)$ denote another integral of $f(x)$ in (a, b). Then

$$\frac{d}{dx}[G(x)-F(x)] = f(x)-f(x)=0$$

Hence in (a, b) the function $G(x)-F(x)$ has a zero derivative and is therefore a constant, § 98. Let c denote this constant. Then $G(x)-F(x)=c$ and therefore $G(x)=F(x)+c$.

Hence if $F(x)$ be any *particular integral* of $f(x)$, the *general integral* of $f(x)$ is $F(x)+C$, where C denotes an arbitrary constant. A general integral is also called an *indefinite integral.* Thus the general integral of $2x$ is x^2+C.

The symbol for the general integral of $f(x)$ or $f(x)dx$ is

$$\int f(x)dx \tag{1}$$

read "integral $f(x)dx$." Hence if $F(x)$ be a particular integral, then

$$\int f(x)dx = F(x)+C \tag{2}$$

To *integrate* $f(x)$ is to express $\int f(x)dx$ in terms of known functions. We may interpret $\int$ as the symbol for this opera-

tion and call $f(x)$ the *integrand* to which it is applied. By the definition of $\int f(x)dx$,

$$\frac{d}{dx}\int f(x)dx = f(x) \qquad \text{or} \qquad d\int f(x)dx = f(x)dx \quad (3)$$

Hence integration is the *inverse* of differentiation, that is, it is the operation which differentiation undoes. To effect it, one must express $f(x)dx$ in terms of differentials whose integrals, like that of $2\,xdx$, are known from the formulas of differentiation.[1]

105. Two properties of integrals. From the definition of integral it follows that if u and v denote functions of x, and k a constant, then

$$\int (u+v)dx = \int u\,dx + \int v\,dx + C \quad (1)$$

$$\int ku\,dx = k\int u\,dx + C \quad (2)$$

For $\int (u+v)dx$ and $\int u\,dx + \int v\,dx$ have the same derivative, $u+v$; and $\int ku\,dx$ and $k\int u\,dx$ have the same derivative, ku.

We may extend (1) to a sum of any finite number of functions.

By (2) it is allowable to shift a constant factor k from one side to the other of the integral sign; but not so a variable factor.

EXAMPLE. Thus

$$\int (x^2+3\,x+5)dx = \int x^2\,dx + \int 3\,x\,dx + \int 5\,dx = \frac{x^3}{3} + \frac{3}{2}x^2 + 5\,x + C$$

106. The integral $\int x^n\,dx$. 1. Let n denote any constant except -1. Since

$$\frac{d}{dx}\frac{x^{n+1}}{n+1} = x^n, \qquad \text{we have} \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (1)$$

[1] Not every $f(x)dx$ can be so expressed. Thus $(1+x^3)^{1/2}dx$ cannot. Hence, though $\int (1+x^3)^{1/2}dx$ exists (in a sense explained in § 126), it cannot be expressed in terms of rational, irrational, trigonometric, exponential, and logarithmic functions.

To integrate x^n, $n \neq -1$, increase the exponent by 1 and then divide by the new exponent.

EXAMPLE 1. Thus, omitting the arbitrary constant C, we have

$$\int x^5\,dx = \frac{x^6}{6},\ \int x^{3/4}\,dx = \tfrac{4}{7}x^{7/4},\ \int \sqrt{x}\,dx = \int x^{1/2}dx = \tfrac{2}{3}x^{3/2},$$

$$\int \frac{dx}{x^3} = \int x^{-3}\,dx = -\frac{x^{-2}}{2} = -\frac{1}{2x^2}.$$

2. Let $n = -1$. Since

$$\frac{d}{dx}\log x = \frac{1}{x}, \qquad \text{we have} \qquad \int \frac{dx}{x} = \log x + C \qquad (2)$$

EXAMPLE 2. Find $\int y^{-1}\,dy$, $\int y^{-1/3}\,dy$, $\int t^8\,dt$, $\int \sqrt[3]{x^5}\,dx$, $\int (1/x^4)\,dx$, $\int (1/\sqrt{x})\,dx$.

The usefulness of (1) and (2) is greatly increased by the following theorem:

107. Theorem. *Let u be a function of x, and $\phi(u)$ a function of u; then*

$$\int \phi(u)\frac{du}{dx}\,dx = \int \phi(u)\,du + C \qquad (1)$$

For the integrals in (1) have the same x-derivative.

Thus
$$\frac{d}{dx}\int \phi(u)\frac{du}{dx}\,dx = \phi(u)\frac{du}{dx}$$

and
$$\frac{d}{dx}\int \phi(u)\,du = \left[\frac{d}{du}\int \phi(u)\,du\right]\frac{du}{dx} = \phi(u)\frac{du}{dx}$$

Therefore under the integral sign, as elsewhere, when u is a function of x we may replace $(du/dx)\,dx$ by du, and vice versa.

EXAMPLE 1. Find $\int 2x\sqrt{x^2+1}\,dx$.

The factor $\sqrt{x^2+1}$ is a function of $u = x^2 + 1$, and the other factor, $2x$, is du/dx. Hence

$$\int 2x\sqrt{x^2+1}\,dx = \int u^{1/2}\frac{du}{dx}\,dx = \int u^{1/2}\,du = \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(x^2+1)^{3/2} + C.$$

Or more briefly, since $2x\,dx = d(x^2+1)$ we have at once by the theorem,

$$\int 2x\sqrt{x^2+1}\,dx = \int (x^2+1)^{1/2}d(x^2+1) = \tfrac{2}{3}(x^2+1)^{3/2} + C$$

EXAMPLE 2. Find $\int \frac{x+1}{x^2+2x}\,dx$

Since $d(x^2+2x)=2(x+1)\,dx$, we may replace $(x+1)dx$ by $d(x^2+2x)$ if at the same time we put the "compensating factor" 1/2 before the integral. Hence

$$\int \frac{x+1}{x^2+2x}\,dx = \frac{1}{2}\int \frac{d(x^2+2x)}{x^2+2x} = \frac{1}{2}\log(x^2+2x)+C$$

EXAMPLE 3. Find $\int (x^3+5)^{2/3}x^2\,dx$, $\int \frac{x\,dx}{1-x^2}$, $\int \frac{x\,dx}{\sqrt{1-x^2}}$,

$$\int \frac{x^2+2x}{x^3+3x^2+1}\,dx$$

In particular, since $d(ax+b)=a\,dx$, we have

$$\int \phi(ax+b)dx = \frac{1}{a}\int \phi(ax+b)\,d(ax+b). \qquad (2)$$

Thus
$$\int \frac{dx}{x+1} = \int \frac{d(x+1)}{x+1} = \log(x+1)+C$$

$$\int \sqrt{1-2x}\,dx = -\frac{1}{2}\int (1-2x)^{1/2}\,d(1-2x) = -\tfrac{1}{3}(1-2x)^{3/2}+C.$$

EXAMPLE 4. Find $\int \sqrt{4x+3}\,dx$, $\int \frac{dx}{\sqrt[3]{2-5x}}$, $\int (2x+7)^{10}\,dx$,

$$\int \frac{\log^3(2x)}{2x}\,dx,\quad \int \frac{x^3+5x}{x-1}\,dx\quad \left[\frac{x^3+5x}{x-1} = x^2+x+6+\frac{6}{x-1}\right]$$

108. Integration by substitution. An integral $\int\phi(x)\,dx$ may often be reduced to an integrable form by expressing x as a function of another variable t. It follows from the theorem of § 107 that if $x=f(t)$, the substitution $x=f(t)$, $dx=f'(t)\,dt$ will transform $\int\phi(x)\,dx$ into an equivalent integral in t. We integrate this t-integral and then express t in terms of x in the result.

EXAMPLE 1. Find $\int \frac{dx}{x^{1/2}+1}$.

Set $x=t^2$ and therefore $dx=2t\,dt$. Then

$$\int \frac{dx}{x^{1/2}+1} = 2\int \frac{t}{t+1}\,dt = 2\int \left[1-\frac{1}{t+1}\right]dt$$
$$= 2t - 2\log(t+1)+C = 2x^{1/2}-2\log(x^{1/2}+1)+C.$$

When $\phi(x)$ is rational with respect to x and $(bx+c)^{1/q}$, the substitution $bx+c=t^q$ will transform $\int\phi(x)\,dx$ into a t-integral whose integrand is rational with respect to t.

EXAMPLE 2. Find $\int\frac{dx}{x^{1/2}-1}$, $\int\frac{x\,dx}{(x+1)^{1/2}}$, $\int\frac{dx}{2+x^{1/3}}$.

EXERCISE XVII

Integrate the following:

1. $\int(x^2+ax+a^2)dx$
2. $\int x^3(2x^2+5)dx$
3. $\int(x^{1/2}+a^{1/2})^2dx$
4. $\int\left(x+\frac{1}{x}\right)^2dx$
5. $\int\frac{dx}{2x-1}$
6. $\int\frac{dx}{(x-1)^2}$
7. $\int\frac{1+x}{x^2}dx$
8. $\int\frac{dx}{\sqrt{4-5x}}$
9. $\int\frac{(x-3)(2x-1)}{x}dx$
10. $\int\frac{1+x}{1-x}dx$
11. $\int(x^4+a^4)^{1/2}x^3\,dx$
12. $\int(ax^2+b)^{2/3}x\,dx$
13. $\int\frac{1+x}{\sqrt{x}}dx$
14. $\int\frac{x^2\,dx}{x^3+2}$
15. $\int\frac{x^2\,dx}{\sqrt{x^3+2}}$
16. $\int\frac{(1+\log x)^{1/2}}{x}dx$
17. $\int\frac{(x-1)(x+3)}{x+2}dx$
18. $\int\frac{x^3}{x-3}dx$
19. $\int(bx+c)^{1/3}dx$
20. $\int\sin^2 x\cos x\,dx$
21. $\int\frac{x\,dx}{(x^2+2)^3}$
22. $\int\frac{x^2\,dx}{(x+1)^{1/2}}$
23. $\int x(2x+5)^{1/2}\,dx$
24. $\int x^3(x-1)^{1/2}\,dx$

109. Determination of a function from its derivative. If y be such a function of x that $dy/dx=f(x)$, or $dy=f(x)dx$, and if $F(x)$ be some particular integral of $f(x)$, then, § 104, $y=F(x)+C$. Hence, when the derivative of a function is known, the function itself is determined except for an arbitrary added constant. If the value of the function for any one value of x is also known, we can find the constant C and

so make the determination of the function complete. Many important problems of geometry and mechanics reduce to the determination of a function from its derivative. In dealing with such a problem one must bear in mind that, when nothing but the derivative is given, the solution is not a single function $y = F(x)$ but the infinite set or "family" of functions $y = F(x) + C$.

EXAMPLE 1. The rate of change of y with respect to x is x^2, and $y = 6$ when $x = 3$; find y when $x = 6$.

Since $dy/dx = x^2$, we have $y = x^3/3 + C$. But since $y = 6$ when $x = 3$, C must be such that $6 = 27/3 + C$, which gives $C = -3$. Hence $y = x^3/3 - 3$, and when $x = 6$, then $y = 69$.

EXAMPLE 2. Find the curves which have the property that the slope at the point whose abscissa is x is $2x - 1$; also that one of the curves which passes through (1, 1).

Since $dy/dx = 2x - 1$, we have $y = x^2 - x + C$ which represents a family of parabolas all having the line $2x - 1 = 0$ for axis.

For the curve through (1, 1), $1 = 1 - 1 + C$ $\therefore C = 1$.

Hence the equation of this curve is

$$y = x^2 - x + 1.$$

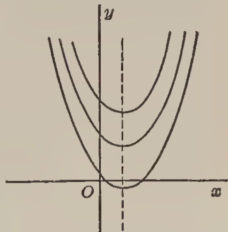

FIG. 77.

EXAMPLE 3. A point is moving on a line with the acceleration $6t$. When $t = 1$ and 2, its distances from the origin are 10 and 80. Find its velocity when $t = 3$.

$$\text{Here } \frac{d^2s}{dt^2} = 6t \qquad \therefore \frac{ds}{dt} = 3t^2 + C_1 \qquad \therefore s = t^3 + C_1 t + C_2.$$

Subst'g the given values of s, $10 = 1 + C_1 + C_2$ and $80 = 8 + 2C_1 + C_2$; $\therefore C_1 = 63$, $C_2 = -54$. Hence the equation of motion is $s = t^3 + 63t - 54$. Also $v = 3t^2 + 63$ $\therefore$ when $t = 3$, $v = 90$.

EXAMPLE 4. Consider the path of a projectile P whose initial velocity is v_0 ft./sec. in a direction which makes the angle θ with the horizontal, the resistance of the air being disregarded. Taking the starting point as origin, and the horizontal and vertical as Ox and Oy, we have $d^2x/dt^2 = 0$, $d^2y/dt^2 = -32$ ft./sec.^{2}; also when $t = 0$, then $x = 0$, $y = 0$, $dx/dt = v_0 \cos\theta$, $dy/dt = v_0 \sin\theta$. Show that the equations of the path are

$$x = v_0 t \cos\theta \qquad y = v_0 t \sin\theta - 16t^2 \tag{1}$$

110. Theorem. *If y be such a function of x that*

$$\phi(x)dx = \psi(y)dy, \quad \textit{then} \quad \int\phi(x)dx = \int\psi(y)dy + C$$

For since $\phi(x) = \psi(y)dy/dx$, we have by the theorem of § 107,

$$\int\phi(x)dx = \int\psi(y)\frac{dy}{dx}dx = \int\psi(y)dy + C$$

This theorem is applicable to any "differential equation" of the form $X_1Y_1dx + X_2Y_2dy = 0$, where X_1, X_2 denote functions of x only, and Y_1, Y_2 functions of y only. For when divided by Y_1X_2 the equation becomes

$$(X_1/X_2)dx + (Y_2/Y_1)dy = 0,$$

which is of the form $\phi(x)dx - \psi(y)dy = 0$.

Example 1. "Integrate" the equation

$$x(y^2 - 1)dx - y(x^2 + 1)dy = 0$$

Dividing by $(y^2 - 1)(x^2 + 1)$, $\quad \dfrac{x}{x^2 + 1}dx - \dfrac{y}{y^2 - 1}dy = 0$

Hence $\quad \displaystyle\int\frac{x\,dx}{x^2 + 1} - \int\frac{y\,dy}{y^2 - 1} = C$

$\therefore \log(x^2 + 1) - \log(y^2 - 1) = 2C \quad \therefore x^2 + 1 = C'(y^2 - 1)$.

Example 2. In Fig. 78, PT and PN are the tangent and normal to the curve S at the point $P(x, y)$. Since

$\dfrac{dy}{dx} = \tan DTP = \tan DPN,$

1. *Subtangent* $DT = ydx/dy$
2. *Subnormal* $DN = ydy/dx$

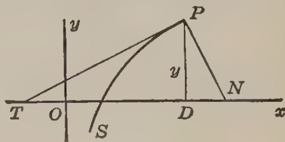

Fig. 78.

Find the curves whose subnormal is the constant $2a$.

We have $y\,dy/dx = 2a \quad \therefore y\,dy = 2a\,dx \quad \therefore y^2/2 = 2ax + C$ or $y^2 = 4ax + C'$.

Example 3. An *orthogonal trajectory* of a set of curves is a curve which cuts all the curves of the set at right angles. Find the orthogonal trajectories of the set of parabolas $y = Cx^2$.

Differentiating $y = Cx^2$ (1) gives $dy/dx = 2Cx$ (2), and eliminating C between (1) and (2) gives $dy/dx = 2y/x$ (3). Hence the curve of the set (1) through the point $P(x, y)$ has at P the slope $2y/x$ $\therefore$ the orthogonal trajectory through P has the slope $-x/2y$. Therefore the differential equation of the set of orthogonal trajectories is dy/dx

$= -x/2\,y$ or $x\,dx + 2\,y\,dy = 0$ (4). The solution of (4) is $x^2/2 + y^2 = C$, which represents a family of ellipses.

EXAMPLE 4. To find y, a function of x whose rate of change with respect to x is proportional to y itself: so that $dy/dx = ky$, where k is a given constant.

$$dy/y = k\,dx \qquad \therefore \log y = kx + C' \qquad \therefore y = e^{kx+C'} = Ce^{kx}$$

If $y = y_0$ when $x = 0$, then $C = y_0$, and we have $\qquad y = y_0 e^{kx}$.

EXERCISE XVIII

1. Find that curve of the set defined by $d^2y/dx^2 = 6\,x$ which passes through the points (1, 1) and (2, 4).

2. Find the curve which passes through the point (2, 3) and whose slope at every point (x, y) on it is (a) x^2 (b) $(1 - x)(1 + y)$ (c) $-y/x$.

3. Find the orthogonal trajectories of the following families of curves:

(a) $y = Cx$ (b) $y^2 = Cx$ (c) $xy = C$ (d) $x^2 - y^2 = C$

4. Find the set of curves for which the subtangent at any point is $x^{1/2}y^{1/2}$.

5. Find that curve through (2, 3) whose subnormal at any point is x/y.

6. For the motion of a body which is sliding down a certain inclined plane we have $ds/dt = 4\,s^{1/2}$. Find the acceleration of the motion, and the inclination of the plane.

7. Assuming that the time rate of decomposition of a quantity y of radium is .038 y, the time unit being 100 years, show how long it would take for one half of a given quantity y_0 to be dissipated.

8. Assuming that the atmospheric pressure as indicated by the barometric reading, p in., decreases as the altitude, h ft., increases, at the rate $dp/dh = -kp$, and that $p = 30$ when $h = 0$ and $p = 28.87$ when $h = 1000$, find h when $p = 25$.

111. Formulas of Integration. Every formula of differentiation, when inverted, yields a formula of integration. The student should familiarize himself with the following. In each of them u may represent the independent variable or a function of that variable; and in each, the constant of integration is to be added on the right.

1. $\int u^n du = \frac{u^{n+1}}{n+1} \quad (n \neq -1)$

2. $\int \frac{du}{u} = \log u$

3. $\int e^u du = e^u$

4. $\int \cos u \, du = \sin u$

5. $\int \sin u \, du = -\cos u$

6. $\int \sec^2 u \, du = \tan u$

7. $\int \operatorname{cosec}^2 u \, du = -\cot u$

8. $\int \sec u \tan u \, du = \sec u$

9. $\int \operatorname{cosec} u \cot u \, du = -\operatorname{cosec} u$

10. $\int \tan u \, du = -\log \cos u$

11. $\int \cot u \, du = \log \sin u$

12. $\int \sec u \, du = \log (\sec u + \tan u)$

13. $\int \operatorname{cosec} u \, du = \log (\operatorname{cosec} u - \cot u)$

14. $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a}$

15. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \log \frac{u-a}{u+a}$

16. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a}$

17. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log (u + \sqrt{u^2 \pm a^2})$

To prove these formulas, it is only necessary to show for each that the derivative of the right member is the integrand in the left member. This is obvious by inspection in the case of 1–9. We shall verify 12. and 17. The reader should verify the others.[1]

[1] When $u < a$, we may replace 15. by $\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \log \frac{a+u}{a-u}$.

$$\frac{d}{du}\log(\sec u + \tan u) = \frac{1}{\sec u + \tan u}\frac{d}{du}(\sec u + \tan u)$$
$$= \frac{\sec u \tan u + \sec^2 u}{\sec u + \tan u} = \sec u$$
$$\frac{d}{du}\log(u + \sqrt{u^2 \pm a^2}) = \frac{1}{u + \sqrt{u^2 \pm a^2}}\frac{d}{du}(u + \sqrt{u^2 \pm a^2})$$
$$= \frac{1 + (u^2 \pm a^2)^{-1/2}u}{u + \sqrt{u^2 \pm a^2}} = \frac{1}{\sqrt{u^2 \pm a^2}}$$

The following integrations illustrate the application of these formulas.

1. $\int e^{3x+4}dx = \frac{1}{3}\int e^{3x+4}d(3\,x + 4) = \frac{1}{3}\,e^{3x+4} + C$
2. $\int\left(\sin 5\,x + \cos\frac{x}{5}\right)dx = -\,\frac{\cos 5\,x}{5} + 5\sin\frac{x}{5} + C$
3. $\int\frac{\cos x}{3 + 4\sin x}\,dx = \frac{1}{4}\int\frac{d(3 + 4\sin x)}{3 + 4\sin x} = \frac{1}{4}\log(3 + 4\sin x) + C$
4. $\int\frac{dx}{3\,x^2 + 2} = \frac{1}{3}\int\frac{dx}{x^2 + 2/3} = \frac{1}{3}\sqrt{\frac{3}{2}}\tan^{-1}\sqrt{\frac{3}{2}}\,x + C$
5. $\int\frac{dx}{(3\,x^2 + 2)^{1/2}} = \frac{1}{\sqrt{3}}\int\frac{dx}{(x^2 + 2/3)^{1/2}}$
$$= \frac{1}{\sqrt{3}}\log[x + (x^2 + 2/3)^{1/2}] + C$$

EXERCISE XIX

Perform the following integrations:

1. $\int\sin(2 - x)dx$ **2.** $\int x\sin x^2\,dx$ **3.** $\int\sec^2 4\,x\,dx$

4. $\int\operatorname{cosec} 3\,x\cot 3\,x\,dx$ **5.** $\int(\tan 2\,x + \cot 3\,x)dx$

6. $\int e^{ax+b}dx$ **7.** $\int e^{\cos x}\sin x\,dx$ **8.** $\int[\sqrt{e^x} + 1/\sqrt{e^x}]dx$

9. $\int(1 + e^x)^{1/2}e^x\,dx$ **10.** $\int 2^x\,dx = \int e^{x\log 2}\,dx$

11. $\int(e^x + e^{-x})^2\,dx$ **12.** $\int\cos^2 3\,x\sin 3\,x\,dx$ **13.** $\int\frac{\sec^2 x}{\tan x + 1}\,dx$

14. $\int(\tan x + \cot x)^2\,dx$ **15.** $\int\frac{dx}{3\,x^2 - 2}$

16. $\int \frac{dx}{4x^2+9}$ 17. $\int \frac{dx}{\sqrt{4-x^2}}$ 18. $\int \frac{dx}{\sqrt{4-9x^2}}$

19. $\int \frac{dx}{\sqrt{x^2-4}}$ 20. $\int \frac{x\,dx}{x^4+1}$ 21. $\int \frac{x\,dx}{\sqrt{x^4+1}}$

22. $\int \frac{dx}{1-\sin x} = \int \frac{1+\sin x}{\cos^2 x}dx = \int (\sec^2 x + \sec x \tan x)dx$

23. $\int \frac{dx}{e^x+1} = \int \left[1 - \frac{e^x}{e^x+1}\right]dx$ 24. $\int \frac{dx}{1+\cos x}$

112. Trigonometric integrals. 1. Integrals of the type $\int \sin^m x \cos^n x\,dx$, where *either m or n is an odd positive integer*, may, by aid of the identity $\sin^2 x + \cos^2 x = 1$, be reduced to integrals with respect to $\sin x$ or $\cos x$ which can be found by § 111, Formula 1. or 2.

EXAMPLE 1. Find $\int \sin^5 x\,dx$

Since $\sin^5 x\,dx = \sin^4 x \cdot \sin x\,dx = -(1-\cos^2 x)^2 \cdot d\cos x$, we have

$$\int \sin^5 x\,dx = -\int (1 - 2\cos^2 x + \cos^4 x)\,d\cos x$$
$$= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C.$$

EXAMPLE 2. Find the following:

1. $\int \sin x \cos x\,dx$ 2. $\int \sin^2 x \cos^3 x\,dx$

3. $\int \sin^{-2/3} x \cos x\,dx$ 4. $\int \frac{\sin^3 x}{\cos^2 x}dx$

2. Products of the types *sin mx cos nx*, *sin mx sin nx*, or *cos mx cos nx* may be integrated as in the following example.

EXAMPLE 3. Find $\int \sin 3x \cos 2x\,dx$

Since $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$,
we have for $A = 3x$, $B = 2x$,

$$\int \sin 3x \cos 2x\,dx = \frac{1}{2}\int (\sin 5x + \sin x)dx = -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$$

EXAMPLE 4. Find $\int \sin 5x \cos x\,dx$, $\int \cos 3x \cos 2x\,dx$, $\int \sin 3x \sin 2x\,dx$

3. Integrals of the type $\int \sec^m x \tan^n x\,dx$ may be found *when m is* 0 *or even* by integrating with respect to $\tan x$, and *when n is odd* by integrating with respect to $\sec x$. The like is true of $\int \operatorname{cosec}^m x \cot^n x\,dx$. Thus

$$\int \tan^3 x\,dx = \int \tan x\,(\sec^2 x - 1)dx = \int \tan x \cdot d\tan x - \int \tan x\,dx$$
$$= \frac{\tan^2 x}{2} + \log\cos x + C$$

$$\int \tan^3 x \sec x\,dx = \int (\sec^2 x - 1)\,d\sec x = \frac{\sec^3 x}{3} - \sec x + C$$

EXAMPLE 5. Find $\int \sec^4 x\,dx$, $\int \tan^3 x \sec^4 x dx$, $\int \operatorname{cosec}^3 x \cot x\,dx$

EXAMPLE 6. Prove that $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

113. Integrands with quadratic denominators. 1. Integrals of the types

$$1.\ \int \frac{dx}{x^2 + bx + c} \qquad 2.\ \int \frac{dx}{(\pm x^2 + bx + c)^{1/2}}$$

can be found by Formulas 14–17 of § 111. We complete the square of the x terms in $\pm x^2 + bx + c$, thus

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)$$
$$-x^2 + bx + c = \left(c + \frac{b^2}{4}\right) - \left(x - \frac{b}{2}\right)^2,$$

and then set $u = x + b/2$, or $u = x - b/2$.

EXAMPLE 1. Find $\displaystyle\int \frac{dx}{2x^2 - 6x + 5}$

$2x^2 - 6x + 5 = 2(x^2 - 3x + 5/2) = 2[(x - 3/2)^2 + 1/4]$. Hence

$$\int \frac{dx}{2x^2 - 6x + 5} = \frac{1}{2}\int \frac{d(x - 3/2)}{(x - 3/2)^2 + 1/4} = \tan^{-1}\frac{x - 3/2}{1/2} + C$$
$$= \tan^{-1}(2x - 3) + C.$$

EXAMPLE 2. Find $\displaystyle\int \frac{dx}{(2ax - x^2)^{1/2}}$

$2ax - x^2 = -(x^2 - 2ax) = -(x^2 - 2ax + a^2) + a^2 = a^2 - (x - a)^2$.
Hence

$$\int \frac{dx}{(2ax - x^2)^{1/2}} = \int \frac{d(x - a)}{[a^2 - (x - a)^2]^{1/2}} = \sin^{-1}\frac{x - a}{a} + C.$$

EXAMPLE 3. Integrate the following:

$$1.\ \int \frac{dx}{x^2 + 4x + 5} \qquad 2.\ \int \frac{dx}{(x^2 + 4x + 5)^{1/2}} \qquad 3.\ \int \frac{dx}{(3 + 2x - x^2)^{1/2}}$$

2. Fractions with denominators like those in 1, 1. and 2., but having numerators of the first degree, may be integrated as in the following examples.

EXAMPLE 4. $\displaystyle\int \frac{x+1}{x^2+1}dx = \int \frac{x\,dx}{x^2+1} + \int \frac{dx}{x^2+1}$

$$= \tfrac{1}{2}\log(x^2+1) + \tan^{-1}x + C.$$

EXAMPLE 5. Find $\displaystyle\int \frac{3x+2}{x^2+2x}dx$ and $\displaystyle\int \frac{3x+2}{(x^2+2x)^{1/2}}dx$

First express the numerator $3x+2$ in terms of the derivative of x^2+2x, that is, find the constants A and B such that

$$3x+2 \equiv A(2x+2) + B \equiv 2Ax + (2A+B).$$

We find A and B by equating coefficients of like powers of x in this supposed identity. Thus $2A = 3$ and $2A + B = 2$ $\therefore A = 3/2$, $B = -1$. Hence

$$\int \frac{3x+2}{x^2+2x}dx = \frac{3}{2}\int \frac{d(x^2+2x)}{x^2+2x} - \int \frac{d(x+1)}{(x+1)^2-1}$$

$$= \frac{3}{2}\log(x^2+2x) - \frac{1}{2}\log\frac{x}{x+2} + C.$$

$$\int \frac{3x+2}{(x^2+2x)^{1/2}}dx = \frac{3}{2}\int \frac{d(x^2+2x)}{(x^2+2x)^{1/2}} - \int \frac{d(x+1)}{[(x+1)^2-1]^{1/2}}$$

$$= 3(x^2+2x)^{1/2} - \log[x+1+(x^2+2x)^{1/2}] + C.$$

EXAMPLE 6. Find $\displaystyle\int \frac{4x+7}{x^2+4x+5}dx$ and $\displaystyle\int \frac{4x+7}{[x^2+4x+5]^{1/2}}dx$

EXERCISE XX

Effect the following integrations:

1. $\int \sin(4-5x)dx$
2. $\int (x^e + e^x)dx$
3. $\int \sqrt{\cos x}\sin^3 x\,dx$
4. $\int \cos^3 x/\sin^4 x\,dx$
5. $\int \tan x \sec^5 x\,dx$
6. $\int \tan^4 x\,dx$
7. $\displaystyle\int \sin 10x \cos 5x\,dx$
8. $\displaystyle\int \frac{a+b\sin x}{\cos^2 x}dx$
9. $\displaystyle\int \frac{dx}{\sqrt{a^2-b^2x^2}}$
10. $\displaystyle\int \frac{dx}{x^2-x+1}$
11. $\displaystyle\int \frac{dx}{x^2-8x+12}$
12. $\displaystyle\int \frac{5x+6}{(x^2+4x)^{1/2}}dx$
13. $\displaystyle\int \frac{p+qx}{a^2+b^2x^2}dx$
14. $\displaystyle\int \frac{dx}{\sqrt{3x-x^2}}$
15. $\displaystyle\int \frac{dx}{\sqrt{x(1-x)}}$

16. $\int \frac{3x-4}{2x^2-3x+1}\,dx$

17. $\int \frac{5-4x}{(1+4x-x^2)^{1/2}}\,dx$

18. $\int \frac{1+2x}{\sqrt{4-x^2}}\,dx$

19. $\int \left(\frac{1-x}{1+x}\right)^{1/2} dx = \int \frac{1-x}{\sqrt{1-x^2}}\,dx$

20. $\int \left(\frac{a+x}{x}\right)^{1/2} dx$

21. By aid of the trigonometric identities $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $1 = \cos^2\theta + \sin^2\theta$, show that

$$\int (1+\cos x)^{1/2} dx = 2\sqrt{2}\sin\frac{x}{2}, \qquad \int (1-\cos x)^{1/2} dx = -2\sqrt{2}\cos\frac{x}{2}.$$

114. Integration by parts. Let u, v denote functions of x, and u', v' their derivatives. Since $d(uv)/dx = uv' + vu'$, we have

$$\int uv'\,dx = vu - \int vu'\,dx \tag{1}$$

Hence if v' denote an integrable factor of $f(x)$, or be 1, and if u be the complementary factor, so that $f(x) = uv'$, then the integral $\int f(x)dx$, or $\int uv'\,dx$, can be found when $\int vu'dx$ can be found, the rule being:

First integrate uv' as if u were a constant, then from the result, vu, subtract the integral of vu'.

This process is called *integration by parts.* It is very useful. Since $v'\,dx = dv$, $u'\,dx = du$, we may also write (1) in the form

$$\int u\,dv = vu - \int v\,du \tag{2}$$

Example 1. Find $\int x \tan^{-1} x\,dx$.

The factor x being integrable, we have by the rule just given

$$\int \tan^{-1} x \cdot x\,dx = \frac{x^2}{2}\tan^{-1} x - \int \frac{x^2}{2}\,\frac{1}{x^2+1}\,dx$$
$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + C.$$

Example 2. Find $\int \log x\,dx$.

There being no immediately integrable factor, we take $v' = 1$ and have

$$\int \log x\,dx = x\log x - \int x\cdot\frac{1}{x}\,dx = x\log x - x + C.$$

EXAMPLE 3. Find $\int x^2 \cos x\,dx$.

Since the exponent of x^2 would be increased by integration, take $v' = \cos x$; then

$$\int x^2 \cdot \cos dx = x^2 \sin x - 2\int x \sin x dx$$
$$\int x \cdot \sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

Hence $\int x^2 \cos x\,dx = x^2 \sin x + 2x\cos x - 2\sin x + C.$

EXAMPLE 4. Show that

$$\int (x^2 + a^2)^{1/2}dx = \tfrac{1}{2}\{x(x^2 + a^2)^{1/2} + a^2 \log [x + (x^2 + a^2)^{1/2}]\}.$$

Int'g by parts, $\displaystyle\int (x^2 + a^2)^{1/2}dx = x(x^2 + a^2)^{1/2} - \int \frac{x^2\,dx}{(x^2 + a^2)^{1/2}}$

But $\displaystyle\int \frac{x^2}{(x^2 + a^2)^{1/2}}\,dx = \int \frac{(x^2 + a^2) - a^2}{(x^2 + a^2)^{1/2}}\,dx$

$$= \int (x^2 + a^2)^{1/2}dx - \int \frac{a^2\,dx}{(x^2 + a^2)^{1/2}}$$

Hence $2\int (x^2 + a^2)^{1/2}dx = x(x^2 + a^2)^{1/2} + a^2 \log [x + (x^2 + a^2)^{1/2}].$

EXAMPLE 5. Find **1.** $\int xe^x\,dx.$ **2.** $\int x^2e^x\,dx.$ **3.** $\int x^3e^{3x}\,dx.$ **4.** $\int x^3 \log x\,dx.$ **5.** $\int \tan^{-1} x\,dx.$ **6.** $\int \sin^{-1} x\,dx.$ **7.** $\int x \sin 2x\,dx.$ **8.** $\int e^x \sin x\,dx.$

EXAMPLE 6. Prove that

$$2\int \sec^3 x\,dx = \sec x \tan x + \log (\sec x + \tan x).$$

115. Reduction formulas for $\int \cos^n x\,dx$ and $\int \sin^n x\,dx$. Integrating by parts,

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cdot \frac{d}{dx} \sin x\,dx$$
$$= \sin x \cos^{n-1} x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx$$
$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

Hence, transposing the last term,

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx \qquad (1)$$

By the same method, or by replacing x by $\pi/2 - x$ in (1), we find

$$n\int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx \qquad (2)$$

These formulas enable one to find $\int \cos^n x\,dx$ and $\int \sin^n x\,dx$ when n is any given positive integer. Formula (1) is especially useful.

EXAMPLE 1. Find $\int \cos^4 x\,dx$ and $\int \cos^2 x \sin^2 x\,dx$.

$$4\int \cos^4 x\,dx = \cos^3 x \sin x + 3\int \cos^2 x\,dx$$

$$2\int \cos^2 x\,dx = \cos x \sin x + \int dx = \cos x \sin x + x + C$$

Hence $\quad \int \cos^4 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C$

and $\quad \int \cos^2 x \sin^2 x\,dx = \int \cos^2 x\,dx - \int \cos^4 x\,dx$

$$= -\tfrac{1}{4}\cos^3 x \sin x + \tfrac{1}{8}(\cos x \sin x + x) + C.$$

EXAMPLE 2. Find $\int \cos^6 x\,dx$, $\quad \int \sin^5 x\,dx$, $\quad \int \cos^3 2x\,dx$, $\quad \int \sin^2 x \cos^4 x\,dx$.

116. Trigonometric substitutions. Expressions involving $\sqrt{a^2 - x^2}$ or $\sqrt{x^2 \pm a^2}$ are often most easily integrated by aid of one of the following substitutions:

when $\sqrt{a^2-x^2}$ occurs, set $x=a\sin\theta$ $\therefore$ $\sqrt{a^2-x^2}=a\cos\theta$

when $\sqrt{x^2+a^2}$ occurs, set $x=a\tan\theta$ $\therefore$ $\sqrt{x^2+a^2}=a\sec\theta$

when $\sqrt{x^2-a^2}$ occurs, set $x=a\sec\theta$ $\therefore$ $\sqrt{x^2-a^2}=a\tan\theta$

EXAMPLE 1. Find $\int \sqrt{a^2 - x^2}\,dx$.

Let $x = a\sin\theta$ $\therefore$ $\sqrt{a^2 - x^2} = a\cos\theta$ and $dx = a\cos\theta\,d\theta$. Then

$$\int \sqrt{a^2 - x^2}\,dx = a^2 \int \cos^2\theta\,d\theta$$

$$= a^2\frac{\sin\theta\cos\theta + \theta}{2} + C$$

$$= \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right] + C$$

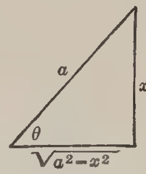

FIG. 79.

EXAMPLE 2. Find $\int \frac{dx}{x\sqrt{x^2 - a^2}}$

Let $x = a\sec\theta$ $\therefore$ $\sqrt{x^2 - a^2} = a\tan\theta$ and $dx = a\sec\theta\tan\theta\,d\theta$.

Then $\quad \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\int d\theta = \frac{\theta}{a} + C = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

EXAMPLE 3. Find $\int \frac{dx}{(x^2 + a^2)^{3/2}}$

Let $x = a \tan \theta \quad \therefore (x^2 + a^2)^{1/2} = a \sec \theta$, and $dx = a \sec^2 \theta \, d\theta$.

Then $\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{a}{a^3} \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta = \frac{1}{a^2} \int \cos \theta \, d\theta$

$$= \frac{1}{a^2} \sin \theta = \frac{1}{a^2} \frac{x}{(x^2 + a^2)^{1/2}}$$

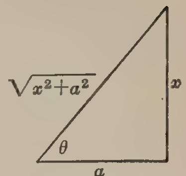

FIG. 80.

This method can be extended to expressions involving $(ax^2 + bx + c)^{1/2}$.

EXAMPLE 4. Find $\int \frac{x\,dx}{\sqrt{4x - x^2}} = \int \frac{x\,dx}{[4 - (x - 2)^2]^{1/2}}$

Let $x - 2 = 2 \sin \theta \quad \therefore \sqrt{4x - x^2} = 2 \cos \theta$ and $dx = 2 \cos \theta \, d\theta$. Then

$$\int \frac{x\,dx}{\sqrt{4x - x^2}} = \int \frac{(2 + 2 \sin \theta) \cdot 2 \cos \theta}{2 \cos \theta} d\theta = 2 \int (1 + \sin \theta) \, d\theta$$

$$= 2\theta - 2 \cos \theta = 2 \sin^{-1} \frac{x - 2}{2} - \sqrt{4x - x^2}$$

EXAMPLE 5. Find 1. $\int \frac{(25 - x^2)^{1/2}}{x} dx$ 2. $\int \frac{dx}{x(3 - x^2)^{1/2}}$

3. $\int (3 - 2x - x^2)^{1/2} dx$ 4. $\int \frac{dx}{(x^2 + 4)^{5/2}}$ 5. $\int \frac{dx}{x(x^2 + 4)^{1/2}}$

6. $\int \frac{(x^2 - 1)^{1/2}}{x} dx$ 7. $\int \frac{dx}{x^3(x^2 - 1)^{1/2}}$.

117. The substitution $x = 1/t$. Integrals of fractions whose denominators contain some power of x or of $x - c$ as a factor may often be simplified by the substitution $x = 1/t$ or $x - c = 1/t$, $dx = -dt/t^2$.

EXAMPLE 1. Setting $x = \frac{1}{t}$ gives

$$\int \frac{dx}{x^2(x^2 + 1)^{1/2}} = -\int \frac{t\,dt}{(1 + t^2)^{1/2}} = -(1 + t^2)^{1/2} = -\frac{(x^2 + 1)^{1/2}}{x}.$$

EXAMPLE 2. Find 1. $\int \frac{dx}{x(x^2 + 1)^{1/2}}$ 2. $\int \frac{dx}{(x - 1)(2 - x^2)^{1/2}}$

3. $\int \frac{dx}{x^2(x^2 + x + 1)^{1/2}}$

EXERCISE XXI

Find the following integrals:

1. $\int x \sin 3x\, dx$
2. $\int x^2 \cos 2x\, dx$
3. $\int x \sec^2 x\, dx$
4. $\int \tan^{-1} \frac{x}{2}\, dx$
5. $\int x^2 \tan^{-1} x\, dx$
6. $\int \tan^{-1} \sqrt{x}\, dx$
7. $\int x \sin^{-1} x\, dx$
8. $\int x^2 \sin^{-1} x\, dx$
9. $\int x^2 e^{4x} dx$
10. $\int x^{-2} \log x\, dx$
11. $\int \log (x^2 - 1) dx$
12. $\int e^{3x} \cos 2x\, dx$
13. $\int (4 - x^2)^{3/2} dx$
14. $\int \frac{dx}{(x^2 + 1)^{5/2}}$
15. $\int \frac{dx}{(x^2 - 1)^{5/2}}$
16. $\int \frac{dx}{(x - 1)(x^2 + 1)^{1/2}}$
17. $\int \frac{dx}{(2 + \tan^2 x)^{1/2}} = \int \frac{\cos x\, dx}{(2 - \sin^2 x)^{1/2}}$
18. $\int \frac{x^2 dx}{(x^2 + 1)^{3/2}}$
19. $\int \frac{dx}{(x^2 + 1)(x^2 + 2)^{1/2}}$, [set $x = \tan \theta$.]
20. Show that setting $n = -m + 2$ in § 115 (1) gives the reduction formula

$$(m - 1) \int \sec^m x\, dx = \sec^{m-1} x \sin x + (m - 2) \int \sec^{m-2} x\, dx$$

118. Integration of rational fractions. Let $f(x)/\phi(x)$ be a proper rational fraction with real coefficients, in its lowest terms. By the fundamental theorem of algebra, § 19, 3., the polynomial $\phi(x)$ is a product of factors of one or both the types $x - a$ and $x^2 + bx + c$, where a, b, c are real but the factors of $x^2 + bx + c$ are imaginary. It can be proved[1] that $f(x)/\phi(x)$ is equal to one and but one sum of fractions, called its *partial fractions*, related as follows to the factors of $\phi(x)$:

1. For each factor $x - a$ occurring but once in $\phi(x)$, there is a single fraction of the form $A/(x - a)$, where A is a real constant, not 0.

2. For each factor $x - a$ occurring r times in $\phi(x)$, there is a group of fractions of the form

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_r}{(x - a)^r}$$

[1] See Fine's College Algebra, §§ 529–536.

where $A_1, \cdots, A_r$ are real constants, any of which except A_r may be 0.

3. For each factor $x^2 + bx + c$ occurring but once in $\phi(x)$, there is a single fraction $(Bx + C)/(x^2 + bx + c)$, where B, C are real constants, not both 0.

4. For each factor $x^2 + bx + c$ occurring r times in $\phi(x)$, there is a group of fractions

$$\frac{B_1x + C_1}{x^2 + bx + c} + \frac{B_2x + C_2}{(x^2 + bx + c)^2} + \cdots + \frac{B_rx + C_r}{(x^2 + bx + c)^r}$$

where $B_1, C_1, \cdots$ are real constants of which not both B_r, C_r are 0.

Thus we may set

$$\frac{x^7 + x^2 - 5}{(2\,x - 1)x^3(3\,x^2 + 1)(x^2 + 1)^2} \equiv \frac{A}{2\,x - 1} + \frac{B_1}{x} + \frac{B_2}{x^2} + \frac{B_3}{x^3} + \frac{Cx + D}{3\,x^2 + 1} + \frac{E_1x + F_1}{x^2 + 1} + \frac{E_2x + F_2}{(x^2 + 1)^2}$$

with assurance that one and but one set of values of the constants A, $B_1, \cdots F_2$ exists for which this identity is true. The values of $A, B_1, \cdots F_2$ may then be found by the methods used in computing undetermined coefficients.

The fractions in 1., 2., 3., 4. can be integrated (see Ex. 4 below). Hence we can integrate any given proper fraction $f(x)/\phi(x)$, if we can find the factors $x - a$ and $x^2 + bx + c$ of $\phi(x)$. To extend the method to an improper fraction, we express it as the sum of an integral expression and a proper fraction.

Example 1. Find $\displaystyle\int \frac{x^2 + 4\,x - 1}{x(2\,x - 1)(x + 2)}dx$

Set $$\frac{x^2 + 4\,x - 1}{x(2\,x - 1)(x + 2)} \equiv \frac{A}{x} + \frac{B}{2\,x - 1} + \frac{C}{x + 2}$$

Then $x^2 + 4\,x - 1 \equiv A(2\,x - 1)(x + 2) + B(x + 2)x + Cx(2\,x - 1)$

Since this is an identity, it is true for all values of x.

Let $x = 0$; then $-1 = A(-1)\cdot 2$ $\therefore A = 1/2$
Let $2\,x - 1 = 0$ or $x = 1/2$; then $5/4 = B(1/2)(5/2)$ $\therefore B = 1$
Let $x + 2 = 0$ or $x = -2$; then $-5 = C\ (-2)(-5)$ $\therefore C = -1/2$

Hence

$$\int \frac{x^2 + 4x - 1}{x(2x-1)(x+2)}dx = \frac{1}{2}\int\frac{dx}{x} + \int\frac{dx}{2x-1} - \frac{1}{2}\int\frac{dx}{x+2}$$
$$= \log\left[\frac{x(2x-1)}{x+2}\right]^{1/2} + C$$

EXAMPLE 2. Find $\int \frac{x^3 + x^2 - 2x - 3}{x^3 - 1}dx$

$$\frac{x^3 + x^2 - 2x - 3}{x^3 - 1} = 1 + \frac{x^2 - 2x - 2}{x^3 - 1} = 1 + \frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)}$$

Set $\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} \equiv \frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1}$

Then $$x^2 - 2x - 2 \equiv A(x^2 + x + 1) + (Bx + C)(x - 1)$$
$$\equiv (A + B)x^2 + (A - B + C)x + (A - C)$$

Equating coefficients of like powers of x, we get

$A + B = 1, A - B + C = -2, A - C = -2 \quad \therefore A = -1, B = 2, C = 1.$

Hence

$$\int\frac{x^3 + x^2 - 2x - 3}{x^3 - 1}dx = \int dx - \int\frac{dx}{x-1} + \int\frac{2x+1}{x^2+x+1}dx$$
$$= x + \log\frac{x^2 + x + 1}{x - 1} + C$$

EXAMPLE 3. Find $\int \frac{x^2 - 4x + 7}{(x-1)^2(x^2+1)}dx$

Set $$\frac{x^2 - 4x + 7}{(x-1)^2(x^2+1)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2 + 1}$$

Then

$$x^2 - 4x + 7 \equiv A(x-1)(x^2+1) + B(x^2+1) + (Cx + D)(x-1)^2 \qquad (1)$$

Let $x = 1$; then $4 = 2B$ and therefore $B = 2$

Transposing the term $2(x^2 + 1)$ thus found to the first member of (1),

$$-x^2 - 4x + 5 \equiv A(x-1)(x^2+1) + (Cx + D)(x-1)^2 \qquad (2)$$

Dividing by $x - 1$,

$$-x - 5 \equiv A(x^2 + 1) + (Cx + D)(x - 1) \qquad (3)$$

Again let $x = 1$. We get $-6 = 2A$ and therefore $A = -3$. Also by equating the coefficients of x^2 and x^0 in (3) we get $A + C = 0$, $A - D = -5 \quad \therefore C = 3, D = 2$. Hence

$$\int \frac{x^2 - 4x + 7}{(x-1)^2(x^2+1)}\,dx$$

$$= -3\int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2} + \int \frac{(3x+2)}{x^2+1}\,dx$$

$$= -3\log(x-1) - \frac{2}{x-1} + \frac{3}{2}\log(x^2+1) + 2\tan^{-1}x + C.$$

EXAMPLE 4. Find $\int \frac{4x-1}{(x^2+2x+5)^2}\,dx.$

Expressing $4x - 1$ in terms of $\frac{d}{dx}(x^2 + 2x - 5)$, as in § 113, Ex. 5, we get

$$\int \frac{4x-1}{(x^2+2x+5)^2}\,dx = 2\int \frac{2x+2}{(x^2+2x+5)^2}\,dx - 5\int \frac{dx}{(x^2+2x+5)^2}$$

$$= -\frac{2}{x^2+2x+5} - 5\int \frac{d(x+1)}{[(x+1)^2+4]^2}$$

The substitution $x + 1 = 2\tan\theta$ gives

$$\int \frac{d(x+1)}{[(x+1)^2+4]^2} = \frac{1}{8}\int \cos^2\theta\,d\theta$$

$$= \tfrac{1}{16}(\cos\theta\sin\theta + \theta)$$

$$= \frac{1}{8}\,\frac{x+1}{x^2+2x+5} + \frac{1}{16}\tan^{-1}\frac{x+1}{2}$$

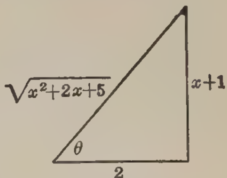

FIG. 81.

Hence

$$\int \frac{4x-1}{(x^2+2x+5)^2}\,dx = -\frac{1}{8}\,\frac{5x+21}{x^2+2x+5} - \frac{5}{16}\tan^{-1}\frac{x+1}{2} + C$$

EXERCISE XXII

Find the following integrals:

1. $\int \frac{x^3}{x^2+3x+2}\,dx$

2. $\int \frac{-4x+6}{(x-1)(x-2)(x-3)}\,dx$

3. $\int \frac{2x+4}{x^3-x}\,dx$

4. $\int \frac{x^2}{(x-a)(x-b)(x-c)}\,dx$

5. $\int \frac{x^2+3x+3}{2x^3+5x^2-3x}\,dx$

6. $\int \frac{x^2+1}{(x+2)^2}\,dx$

7. $\int \frac{2x-1}{x(x-2)^2}\,dx$

8. $\int \frac{3x^2+1}{x(x^2+x+1)}\,dx$

9. $\int \frac{dx}{x^4-16}$

10. $\int \frac{x^2 + 5x + 2}{x^4 - 5x^2 + 4} dx$

11. $\int \frac{2x^3 + 9}{x^4 + x^3 - 12x^2} dx$

12. $\int \frac{dx}{(x^2 - 1)^2}$

13. $\int \frac{x^3 + 4}{(x^2 + 2x + 2)^2} dx$

14. $\int \frac{dx}{(x^2 + 9)^3}$

15. $\int \frac{x^{1/3} - 1}{x^{3/2} - x} dx$ (set $x = t^6$)

119. General conclusions. Let $R(u)$ or $R(u, v)$ denote a rational function of u or of u, v.

1. It is shown in § 118 that every integral of the form $\int R(x)dx$ is expressible in terms of rational, logarithmic and $\tan^{-1}$ functions.

2. Any integral which has the form $\int R(\sin\theta, \cos\theta)d\theta$ may be reduced by the substitution $\theta = 2\tan^{-1} t$ to the form $\int R(t)dt$. For if $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2},\quad \cos\theta = \frac{1-t^2}{1+t^2},\quad d\theta = \frac{2\,dt}{1+t^2} \tag{1}$$

all of which are rational with respect to t.

3. Any integral of the form $\int R[x, (ax^2 + bx + c)^{1/2}]dx$ may be reduced by the trigonometric substitutions of § 116 to the form $\int R(\sin\theta, \cos\theta)d\theta$, and therefore, by 2., to the form $\int R(t)dt$.

Example. By the substitution (1), show that

$$\int \operatorname{cosec} x\, dx = \log\tan\frac{x}{2},\quad \int \sec x\, dx = \log\tan\left[\frac{x}{2} + \frac{\pi}{4}\right].$$

Also find $\int \frac{dx}{2 + \cos x}$ and $\int \frac{dx}{3\sin x + 4\cos x}$

XV. DEFINITE INTEGRALS

120. Second fundamental problem of the calculus. The second principal problem of the calculus is to define and evaluate the numerical measures of continuous magnitudes, as for example geometric surfaces and solids. The method of definition is illustrated by § 121, and the theorems which justify this method are given in §§ 122, 123, 124.

121. Area. Consider the space $R = abBA$ bounded by the lines Ox, $x = a$, $x = b$, and a continuous curve arc AB which does not cross Ox. The elementary definition of area applies to rectilinear figures only. It is desired to extend this notion to the space R.

Suppose the segment ab to be divided and redivided into parts in any manner such that as the process is continued the length of every part approaches the limit 0. At any stage of the process let $h_1, h_2, \cdots, h_n$ denote the parts and also their lengths. On each part h_i as base construct the rectangles whose altitudes are the lengths m_i and M_i of the least and greatest curve ordinates on h_i. The rectangles with least altitudes form a rectilinear figure contained in R, whose area is

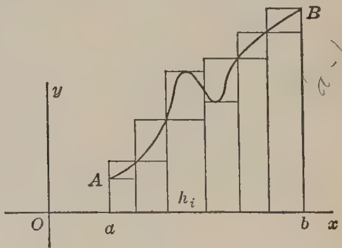

FIG. 82.

$$s_n = m_1h_1 + m_2h_2 + \cdots + m_nh_n$$

The rectangles with greatest altitudes form a rectilinear figure containing R, whose area is

$$S_n = M_1h_1 + M_2h_2 + \cdots + M_nh_n$$

In § 124 it is proved that as $n \to \infty$ the sums s_n and S_n approach one and the same number l as limit. We call this limit l the *area of the space R*.

Let $y = f(x)$ be the equation of the arc AB. Then if x_i be any point in h_i, we shall have $m_i \leqq f(x_i) \leqq M_i$ and therefore

$$\text{area } abBA = \lim_{n\to\infty} [f(x_1)h_1 + f(x_2)h_2 + \cdots + f(x_n)h_n]$$

122. Uniform continuity. Let $f(x)$ be continuous in the interval (a, b); the difference between the greatest and least values of $f(x)$ in any part of (a, b) is called the *oscillation* of $f(x)$ in that part. Thus the oscillation of x^2 in $(1.2, 1.3)$ is $1.3^2 - 1.2^2 = .25$. It will be proved later that

Corresponding to any assigned positive number ϵ there is another positive number δ, such that in every part of (a, b) whose length is less than δ, the oscillation of $f(x)$ is less than ϵ.

To indicate this, we say that if $f(x)$ is continuous in (a, b), it is *uniformly* continuous in (a, b).

123. Greatest and least values in an interval and its parts. Evidently the greatest value of $f(x)$ in a part of an interval cannot exceed its greatest value in the whole interval. Hence if an interval of length h be divided into parts of lengths h_1 and h_2, and if M, M_1, M_2 denote the greatest values of $f(x)$ in the whole interval and these parts, then since $M \geqq M_1, M_2$,

$$\begin{aligned} Mh &= M(h_1 + h_2) \\ &= Mh_1 + Mh_2 \geqq M_1h_1 + M_2h_2 \end{aligned} \tag{1}$$

Similarly, if m, m_1, m_2 denote the least values of $f(x)$ in the whole interval and the parts,

$$mh \leqq m_1h_1 + m_2h_2 \tag{2}$$

The like is true when an interval is divided into any number of parts.

124. Theorem. *Let $f(x)$ be continuous in the interval (a, b). Suppose (a, b) to be divided and redivided into parts in any manner such that as the process is continued the lengths of the parts all approach 0 as limit. At any stage of the process let $h_1, h_2, \cdots, h_n$ denote the parts and also their lengths; also let x_1 denote any point in h_1, x_2 any point in h_2, and so on. Then, as $n \to \infty$, the sum of products*

$$\Sigma f(x_i)h_i = f(x_1)h_1 + f(x_2)h_2 + \cdots + f(x_n)h_n \tag{1}$$

approaches a limit, and the value of this limit is independent of the mode of division of (a, b) into parts h_i and of the choice of the x_i's in the h_i's.

For let M_i and m_i denote the greatest and least values of $f(x)$ in the part h_i, and form the sums

$$S_n = M_1h_1 + M_2h_2 + \cdots + M_nh_n \tag{2}$$

$$s_n = m_1h_1 + m_2h_2 + \cdots + m_nh_n \tag{3}$$

Since $s_n \leqq \Sigma f(x_i)h_i \leqq S_n$, the theorem will be proved if it can be shown that $\lim (S_n - s_n) = 0$ and that S_n approaches the same limit l for all modes of division of (a, b) into parts h_i.

1. *The proof that* $\lim (S_n - s_n) = 0$.

$$S_n - s_n = (M_1 - m_1)h_1 + (M_2 - m_2)h_2 + \cdots + (M_n - m_n)h_n$$

But $M_i - m_i$ is the oscillation of $f(x)$ in the part h_i. Hence, § 122, we can take n great enough to make every $M_i - m_i$ less than any assigned positive number ϵ and therefore to make

$$S_n - s_n < \epsilon h_1 + \epsilon h_2 + \cdots + \epsilon h_n = \epsilon(b - a) \tag{4}$$

We can take $\epsilon(b - a)$ as small as we please; hence

$$\lim (S_n - s_n) = 0$$

2. *The proof that S_n approaches a unique limit l.*

First. Suppose that $h_1, h_2, \cdots h_n$ are obtained by dividing (a, b) into certain parts, then *subdividing* these parts, and so on. Then, as n increases, S_n decreases (or remains un-

changed), § 123 (1), but remains always greater than any value of s_n. Hence in this case S_n approaches a limit l, § 5.

Second. Representing the points which divide (a, b) into the parts $h_1, h_2, \cdots, h_n$ just considered by $t_1, t_2, \cdots, t_{n-1}$, let $z_1, z_2, \cdots, z_{m-1}$ denote any other set of points which divide (a, b) into parts whose lengths all $\rightarrow 0$ when $m \rightarrow \infty$. Let S'_m and s'_m denote the sums like (2) and (3) which correspond to these z-parts. Then $\lim\limits_{m \to \infty} S'_m = \lim\limits_{n \to \infty} S_n = l$.

For let S''_p and s''_p denote the sums like (2) and (3) corresponding to the parts into which the t-points and the z-points when taken together divide (a, b).

As already proved, (4), we can take n and m great enough to make

$$S_n - s_n < \epsilon(b - a) \quad \text{and} \quad S'_m - s'_m < \epsilon(b - a)$$

The z-points subdivide the t-parts; hence $S_n \geqq S''_p \geqq s_n$, and therefore

$$S_n - S''_p \leqq S_n - s_n < \epsilon(b - a) \tag{5}$$

The t-points subdivide the z-parts; hence $S'_m \geqq S''_p \geqq s'_m$, and therefore

$$S'_m - S''_p \leqq S'_m - s'_m < \epsilon(b - a) \tag{6}$$

But $S_n - S'_m = (S_n - S''_p) - (S'_m - S''_p)$; hence

$$|S_n - S'_m| < \epsilon(b - a) \tag{7}$$

We can take $\epsilon(b - a)$ as small as we please; hence

$$\lim S'_m = \lim S_n = l$$

We represent the limit l by the symbol $\lim\, [\Sigma f(x_i)h_i]_a^b$.

125. Properties of $\lim\, [\Sigma f(x_i)h_i]_a^b$. 1. If we replace the interval (a, b) by the interval (b, a), we change the sign of every h_i in the sum $\Sigma f(x_i)h_i$, but leave the sum otherwise unchanged. Hence

$$\lim\, [\Sigma f(x_i)h_i]_b^a = - \lim\, [\Sigma f(x_i)h_i]_a^b \tag{1}$$

2. If we divide the interval (a, b) into two parts, (a, c) and (c, b), we may break up the sum $[\Sigma f(x_i)h_i]_a^b$ into the two sums $[\Sigma f(x_i)h_i]_a^c$ and $[\Sigma f(x_i)h_i]_c^b$. Hence

$$\lim\, [\Sigma f(x_i)h_i]_a^b = \lim\, [\Sigma f(x_i)h_i]_a^c + \lim\, [\Sigma f(x_i)h_i]_c^b \qquad (2)$$

By aid of (1) we can extend (2) to the case when $c < a$ or $c > b$.

3. By § 123 the value of $\lim\, [\Sigma f(x_i)h_i]_a^b$ is between $m(b - a)$ and $M(b - a)$, where m and M denote the least and greatest values of $f(x)$ in (a, b). Hence, § 18, 2., it is equal to the value of $f(x)(b - a)$ for some value $\bar{x}$ of x in (a, b). We have, therefore,

$$\lim\, [\Sigma f(x_i)h_i]_a^b = f(\bar{x})(b - a) \quad [a < \bar{x} < b] \qquad (3)$$

126. The function $F(x) = \lim\, [\Sigma f(x_i)h_i]_a^x$. In $[\Sigma f(x_i)h_i]_a^b$, replace the constant b by a variable x whose values are in (a, b). To each value of x corresponds a definite interval (a, x) and therefore, § 124, a definite number $\lim\, [\Sigma f(x_i)h_i]_a^x$.

Hence $\lim\, [\Sigma f(x_i)h_i]_a^x$ is a function of x in (a, b), § 14. Let $F(x)$ denote this function:

$$F(x) = \lim\, [\Sigma f(x_i)h_i]_a^x \qquad (1)$$

we shall prove that

$$dF(x)/dx = f(x) \qquad (2)$$

For give to x the increment Δx. Then, § 125 (1), (2), (3),

$$\begin{aligned} F(x + \Delta x) - F(x) &= \lim\, [\Sigma f(x_i)h_i]_a^{x+\Delta x} - \lim\, [\Sigma f(x_i)h_i]_a^x \\ &= \lim\, [\Sigma f(x_i)h_i]_x^{x+\Delta x} = f(\bar{x})\Delta x \end{aligned}$$

where $\bar{x}$ denotes some number between x and $x + \Delta x$.

Therefore, since $\bar{x} \to x$ when $\Delta x \to 0$, we have

$$\frac{dF(x)}{dx} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} f(\bar{x}) = f(x)$$

1. Since $dF(x)/dx = f(x)$, the function $F(x)$ is an integral of $f(x)$, § 104. We represent this integral by the symbol $\int_a^x f(x)dx$, read "integral $f(x)dx$, a to x," and call a the *lower limit* and x the *upper limit* of the integral.

Hence, by definition,

$$\int_a^x f(x)dx = \lim\ [\Sigma f(x_i)h_i]_a^x \tag{3}$$

2. Let $\phi(x)$ denote any *known* integral of $f(x)$ in the interval (a, b), that is, any function, found by inspection or the process of integration, whose derivative throughout (a, b) is $f(x)$ and which therefore is continuous in (a, b). From this integral $\phi(x)$, the integral $\int_a^x f(x)dx$ can differ by a constant only, § 104, that is,

$$\int_a^x f(x)dx = \phi(x) + C \tag{4}$$

To find C, let $x \to a$. Then $\int_a^x f(x)dx \to 0$ and $\phi(x) \to \phi(a)$. We therefore have $0 = \phi(a) + C \quad \therefore C = -\phi(a)$. Therefore

$$\int_a^x f(x)dx = \phi(x) - \phi(a) \tag{5}$$

Hence $\int_a^x f(x)dx$ is known when $\phi(x)$ is known.

3. Since $\lim\ [\Sigma f(x_i)h_i]_a^x$ is an integral of $f(x)$, *every continuous function $f(x)$ has integrals.*

127. Definite integrals. Setting $x = b$ in § 126 (3), we have

$$\int_a^b f(x)dx = \lim\ [\Sigma f(x_i)h_i]_a^b \tag{1}$$

The number $\int_a^b f(x)dx$ which this formula defines is called the *definite integral* of $f(x)$ between the limits a and b.

Again, setting $x = b$ in § 126 (5), we have

$$\int_a^b f(x)dx = \phi(b) - \phi(a) \tag{2}$$

If an integral $\phi(x)$ of $f(x)$ in (a, b) is known, then $\int_a^b f(x)dx$ may be found by the rule: From $\phi(b)$ subtract $\phi(a)$.

Example 1. Find $\int_2^5 (2x + 5)dx$

Integrating $2x + 5$, we obtain $\phi(x) = x^2 + 5x$. Hence

$$\int_2^5 (2x + 5)dx = \phi(5) - \phi(2) = 50 - 14 = 36$$

For convenience we use $[\phi(x)]_a^b$ as a symbol for $\phi(b) - \phi(a)$ and write

$$\int_2^5 (2x + 5)dx = [x^2 + 5x]_2^5 = 50 - 14 = 36$$

EXAMPLE 2. The interval (1, 3) is divided into n equal parts, and the length of each part is multiplied by the square of any number in that part. Find the limit, when $n \to \infty$, of the sum of the products thus obtained.

$$\lim\,[\Sigma x_i^2 h_i]_1^3 = \int_1^3 x^2\,dx = \left[\frac{x^3}{3}\right]_1^3 = 9 - \frac{1}{3} = 8\tfrac{2}{3}$$

EXAMPLE 3. Find $\displaystyle\int_{-1}^{1} \frac{dx}{x^2+1}$

The simplest integral of $1/(x^2+1)$ is $\tan^{-1} x$, that is, the angle between $-\pi/2$ and $\pi/2$ whose tangent is x. When x increases continuously from -1 to 1, $\tan^{-1} x$ increases continuously from $-\pi/4$ to $\pi/4$. See Fig. 40, p. 66. Hence

$$\int_{-1}^{1} \frac{dx}{x^2+1} = \tan^{-1}(1) - \tan^{-1}(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2} \qquad (a)$$

Similarly

$$\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3} \qquad (b)$$

EXERCISE XXIII

Evaluate the following definite integrals 1–12:

1. $\int_{-2}^{1} (2x^3 - x^2)dx$ 2. $\int_0^{\pi} \sin\theta\,d\theta$ 3. $\int_0^3 e^{2x-6}dx$

4. $\int_0^{\pi/2} \sin^2\theta\,d\theta$ 5. $\int_{-\pi/4}^{\pi/3} \tan^2\theta\,d\theta$ 6. $\int_0^{\pi/2} x\cos x\,dx$

7. $\displaystyle\int_0^2 \frac{dx}{x^2+4}$ 8. $\displaystyle\int_{-1}^{\sqrt{2}} \frac{dx}{\sqrt{4-x^2}}$ 9. $\displaystyle\int_{-1}^{1} \frac{x\,dx}{x^2+1}$

10. $\int_0^1 x(1-x)^{1/2}\,dx$ 11. $\int_1^4 x^2(1+\sqrt{x})dx$ 12. $\int_{-1}^{1} \tan^{-1} x\,dx$

13. Referring to the formula § 126 (5), show that

(1) $\log x = \displaystyle\int_1^x \frac{dx}{x}$ (2) $\tan^{-1} x = \displaystyle\int_0^x \frac{dx}{x^2+1}$ (3) $\sin^{-1} x = \displaystyle\int_0^x \frac{dx}{\sqrt{1-x^2}}$

14. By the definition $\int_a^b f(x)dx = \lim\,[\Sigma f(x_i)h_i]_a^b$ and § 125, 3, show that

(1) $\displaystyle\int_{-2}^{2} \frac{x\,dx}{x^4+1} = 0$ (2) $\displaystyle\int_0^2 \frac{dx}{(x^3+1)^{1/2}}$ is $> \frac{2}{3}$ and < 2

128. Computation of areas. 1. By the definition of area in § 121, the area of the space $R = abBA$ (Fig. 82) bounded by the lines Ox, $x = a$, $x = b$ and an arc AB of a curve $y = f(x)$ which lies above Ox is $\lim\,[\Sigma f(x_i)h_i]_a^b$.

Hence if A_a^b denote this area, we have, § 127 (1),

$$A_a^b = \int_a^b y\,dx \tag{1}$$

When the arc AB is below Ox, the integral $\int_a^b y\,dx$ is negative; its numerical value is the area of R.

EXAMPLE 1. Find the areas of the spaces bounded by Ox and the curve $y = x^3 - x^2 - 2x$.

Since $x^3 - x^2 - 2x = (x+1)x(x-2)$, the curve cuts Ox at the points $x = -1, 0, 2$, is above Ox in $(-1, 0)$, below in $(0, 2)$.

$$\int_{-1}^0 y\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^0 = \frac{5}{12}$$

$\therefore$ area $ABO = \frac{5}{12}$

$$\int_0^2 y\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_0^2 = -\frac{8}{3}$$

$\therefore$ area $OCD = \frac{8}{3}$

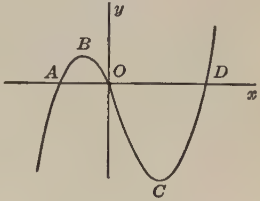

FIG. 83.

Observe that $\int_{-1}^2 y\,dx$ is the *difference* of the areas ABO and OCD.

EXAMPLE 2. Draw the curve $y = x^2 + 2x - 3$ and find the area of the space between it and Ox; also the area of the space bounded by the curve and the lines Ox, $x = 2$, $x = 4$.

EXAMPLE 3. Show that if $f(x)$ is $+$ between $x = a$ and $x = b$, the area of the space bounded by the curve $y^2 = f(x)$ and the lines $x = a$ and $x = b$ is $2\int_a^b [f(x)]^{1/2} dx$.

Find the area of the space between the parabola $y^2 = 4x$ and the line $x = 9$.

2. Suppose that the curve $y_1 = f(x)$ lies above the curve $y_2 = \phi(x)$ throughout the x-interval (a, b). Then the area of the space $ABCDEF$ bounded by the two curves and the lines $x = a$, $x = b$ is

$$A_a^b = \int_a^b (y_1 - y_2)\,dx \tag{2}$$

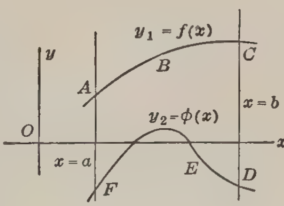

FIG. 84.

This follows from 1. and the identity

$$\int_a^b (y_1 - y_2)\,dx = \int_a^b y_1\,dx - \int_a^b y_2\,dx$$

EXAMPLE 4. Find area of the triangle whose sides are $y = 2x$ (1), $x + y = 6$ (2), $y = x$ (3).

(1), (2) meet at A (2, 4); (2), (3) meet at B (3, 3).

$OCA = \int_0^2 (2x - x)\,dx = \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = 2$

$ACB = \int_2^3 (6 - x - x)\,dx = \int_2^3 (6 - 2x)\,dx = 1$

Hence $OBA = 2 + 1 = 3$

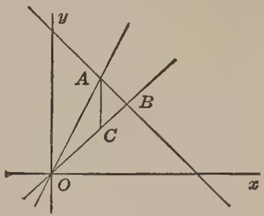

FIG. 85.

EXAMPLE 5. Find area bounded by the parabola $(y - x)^2 = 4x$ and the line $x = 5$.

To find the curve, we proceed as in § 92, Ex. 2, and solving for y obtain

$y_1 = x + 2x^{1/2}(OA)$ and $y_2 = x - 2x^{1/2}(OB)$

$\therefore OAB = \int_0^5 (y_1 - y_2)dx = 4\int_0^5 x^{1/2}dx = \frac{40}{3}\sqrt{5}.$

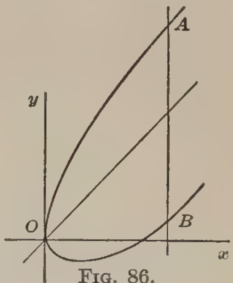

FIG. 86.

EXERCISE XXIV

Find the areas bounded by the following curves, 1–9, and Ox, drawing the curves.

1. $y = 9 - x^2$ **2.** $y = x^2 - 3x - 4$ **3.** $y = 3x^2 - x^3$

4. $y = x(2x - 5)^2$ **5.** $y = x^3 + 2x^2 - 3x$ **6.** $y = (x^2 - 1)^2$

7. $(y - 2x)^2 = 4x$ **8.** $(y - x)^2 = x + 2$ **9.** $y = (1 - x^2)/(x + 2)$

By the definition in § 121, the area bounded by the curve $x = \phi(y)$ and the lines Oy, $y = c$, $y = d$, when $\phi(y)$ is + in the y-interval (c, d), is $\int_c^d \phi(y)dy$. Find the area bounded by each of the following curves, 10–12, and Oy.

10. $x = 4y - y^2$ **11.** $y^2 - 4x - 4 = 0$ **12.** $y^2 - 4x - 4y = 0$

13. Find in two ways the area bounded by the curve $y = x^3$, Oy, and the line $y = 8$.

Find the areas bounded by the following lines or curves, 14–21, drawing the graphs.

14. $x - 2y + 4 = 0$, $x = 0$, $y = 3$, $y = 5$

15. $x - 2y + 6 = 0$, $x + y = 0$, $x - y = 0$

16. $y = x^2,\ y = x,\ y = 2x$

17. $y^2 = 4ax,\ y = 2x$

18. $y^2 = 4ax,\ x^2 = 4ay$

19. $y = x^2,\ x - y + 2 = 0$

20. $y = x^2 + x,\ y = 2x^2 - 2$

21. $y = x^3 - x,\ y = 3x - x^3$

22. Find the areas bounded by the loops of each of the curves:

(1) $y^2 = x^2(a - x)$ (2) $y^2 = x^2(a^2 - x^2)$

23. Find the area bounded by the hyperbola $xy = 6$ and the lines $y = 0$, $x = 1$, $x = e^2$.

24. Show that the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the lines $x = 0$ and $x = x$ is $(b/a)[x(a^2 - x^2)^{1/2} + a^2 \sin^{-1} x/a]$; hence that the area of the ellipse itself is $ab\pi$. See § 116, Ex. 1.

25. Show that the area bounded by the hyperbola $x^2/a^2 - y^2/b^2 = 1$ and the line $x = x$ is $(b/a)[x(x^2 - a^2)^{1/2} - a^2 \log \{x + (x^2 - a^2)^{1/2}\}]$. See § 114, Ex. 4.

26. Find the area bounded by the catenary $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$ and the lines $y = 0$, $x = 0$, $x = a$.

27. Find the area bounded by the parabola $x^{1/2} + y^{1/2} = a^{1/2}$ and the lines $x = 0$, $y = 0$.

129. Properties of definite integrals. For convenience represent $\Sigma f(x_i)h_i$ by $\Sigma f(x)\delta x$, where δx stands for any one of the parts h_i into which the given x-interval (a, b) is supposed to be divided, and x for any point x in that part δx. We call $f(x)\delta x$ an *element* of $\Sigma f(x)\delta x$. We then have

$$\int_a^b f(x)dx = \lim\,[\Sigma f(x)\delta x]_a^b \tag{1}$$

1. By (1), the value of $\int_a^b f(x)dx$ depends solely on a, b, and the form of $f(x)$. Hence

$$\int_a^b f(x)dx = \int_a^b f(y)dy \tag{2}$$

2. The formulas (1) and (2) of § 125, give

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \tag{3}$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx \tag{4}$$

3. If m and M denote the least and greatest values of $f(x)$ in (a, b), then, § 125, 3.,

$$m(b - a) \leqq \int_a^b f(x)dx \leqq M(b - a) \tag{5}$$

and there is therefore a value $\bar{x}$ of x in (a, b) such that

$$\int_a^b f(x)dx = f(\bar{x})(b - a) \tag{6}$$

For the $y = f(x)$ whose graph in (a, b) is the arc APB in Fig. 87 we have $m = aA$, $M = bB$ and the geometric meaning of (5) is that in area

$$abCA < abBPA < abBD$$

and by (6) there is a point $\bar{x}$ in (a, b) such that the rectangle aF of altitude $\bar{x}P = f(\bar{x})$ equals $abBPA$ in area.

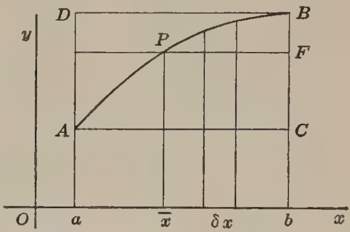

FIG. 87.

EXAMPLE 1. Show that $\int_2^4 \frac{dx}{x^3 + 1}$ is between $\frac{2}{65}$ and $\frac{2}{9}$. Locate its value more closely by dividing the interval (2, 4) into (2, 3) and (3, 4) and then applying (4) and (5).

EXAMPLE 2. Show that $\int_1^3 (x^2 - 4x + 5)^{1/2}dx$ is between 2 and $2\sqrt{2}$.

EXAMPLE 3. Let ΔS denote the area of the strip on δx in Fig. 87; show, by (6), that there is a point x' in δx such that $\Delta S = f(x')\delta x$.

EXAMPLE 4. Divide (a, b) into the parts, $h_1, h_2, \ldots, h_n$, and show by (4) and (6) that points exist, x_1 in h_1, x_2 in h_2, ..., x_n in h_n, such that

$$\int_a^b f(x)dx = f(x_1)h_1 + f(x_2)h_2 + \ldots + f(x_n)h_n.$$

EXAMPLE 5. Show that (6) is the mean value theorem, § 97, for the function $F(x) = \int_a^x f(x)dx$.

EXAMPLE 6. By (1) and § 4, (3), show that

$$\left| \int_a^b f(x)dx \right| \leqq \int_a^b |f(x)|\, dx$$

EXAMPLE 7. Show that if $\phi(x) > 0$ throughout (a, b), and m and M are the least and greatest values of $f(x)$ in (a, b), then

$$m\int_a^b \phi(x)dx \leqq \int_a^b f(x)\phi(x)dx \leqq M\int_a^b \phi(x)dx$$

130. Related integrals. 1. Suppose each of two equal x-intervals, (a, b) and (c, d), to be divided into n equal parts δx. If, whatever the value of n, the sums $[\Sigma f(x)\delta x]_a^b$ and $[\Sigma f(x)\delta x]_c^d$ are equal, term to term, then $\int_a^b f(x)dx = \int_c^d f(x)dx$. If the terms of the first sum are the negatives of those of the second, then $\int_a^b f(x)dx = -\int_c^d f(x)dx$.

Hence in particular, by separating $(-a, a)$ into $(-a, 0)$ and $(0, a)$,

If $f(-x) \equiv f(x)$, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ (1)

If $f(-x) \equiv -f(x)$, then $\int_{-a}^a f(x)dx = 0$ (2)

EXAMPLE 1. Prove the following relations and interpret them geometrically.

$$\int_0^\pi \sin x\,dx = 2\int_0^{\pi/2} \sin x\,dx, \quad \int_{-\pi/2}^{\pi/2} \sin x\,dx = 0$$

$$\int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\int_0^{\pi/2} \cos x\,dx, \quad \int_0^\pi \cos x\,dx = 0$$

EXAMPLE 2. Prove the following:

1. $\int_0^\pi \sin^m x \cos^n x\,dx = 0$ when n is odd, $\int_{-\pi/2}^{\pi/2} \sin^m x \cos^n x\,dx = 0$ when m is odd.

2. $\int_0^\pi (\sin x + \cos x)^2 dx = \pi$ 3. $\displaystyle\int_{-1}^1 \frac{x^3 + x^2}{x^4 + 1}dx = 2\int_0^1 \frac{x^2\,dx}{x^4 + 1}$.

2. If the interval $(0, a)$ be divided into n equal parts δx, the terms of the sum $[\Sigma f(x)\delta x]_0^a$ are the same as those of $[\Sigma f(a - x)\delta x]_0^a$ in reverse order; hence

$$\int_0^a f(x)dx = \int_0^a f(a - x)dx \tag{3}$$

Thus, $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^n(\pi/2 - x)dx = \int_0^{\pi/2} \cos^n x\,dx$ (4)

131. The integrals $\int_0^{\pi/2} \sin^n x\,dx$, $\int_0^{\pi/2} \cos^n x\,dx$. Let n be any positive integer. By § 115, (1),

$$n\int_0^{\pi/2} \cos^n x\,dx = [\cos^{n-1} x \sin x]_0^{\pi/2} + (n - 1)\int_0^{\pi/2} \cos^{n-2} x\,dx$$

But $[\cos^{n-1} x \sin x]_0^{\pi/2} = 0$; for $\cos \pi/2 = 0$ and $\sin 0 = 0$. Hence

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx$$

By repeated applications of this formula we find

1. For n even, $$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} \qquad (1)$$

2. For n odd, $$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3} \qquad (2)$$

These formulas apply also to $\int_0^{\pi/2}\sin^n x\,dx$, § 130 (4).

Thus, $\int_0^{\pi/2}\cos^4 x\,dx = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{16}\pi$, $\int_0^{\pi/2}\sin^5 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$,

$$\int_0^{\pi}\sin^2 x\cos^4 x\,dx = 2\int_0^{\pi/2}(\cos^4 x - \cos^6 x)dx$$
$$= 2\left[\frac{3}{4}\,\frac{1}{2} - \frac{5}{6}\,\frac{3}{4}\,\frac{1}{2}\right]\frac{\pi}{2} = \frac{\pi}{16}$$

132. Substitutions. The value of $\int_a^b f(x)dx$ is sometimes found most readily by aid of a substitution $x = \phi(t)$. Let t_1 and t_2 be values of t such that (1) $a = \phi(t_1)$ and $b = \phi(t_2)$, (2) when t changes continuously from t_1 to t_2, x increases continuously from a to b, (3) $\phi'(t)$ is continuous in (t_1, t_2). Then

$$\int_a^b f(x)dx = \int_{t_1}^{t_2} f[\phi(t)]\phi'(t)dt \qquad (1)$$

For let δt denote any part of the t-interval (t_1, t_2), and δx the corresponding part of the x-interval (a, b). By the mean value theorem, § 97 (6),

$$\delta x = \phi'(t_i)\delta t$$

where t_i is some number in δt. Hence, if $x_i = \phi(t_i)$,

$$f(x_i)\delta x = f[\phi(t_i)]\phi'(t_i)\delta t \qquad (2)$$

where x_i is in δx. The members of (2) are corresponding elements of the integrals in (1). Hence these integrals are equal.

Under similar conditions, if $x = \phi(t)$, $y = \psi(t)$, then

$$\int_a^b f(x, y)dx = \int_{t_1}^{t_2} f[\phi(t), \psi(t)]\phi'(t)dt \qquad (3)$$

EXAMPLE 1. Find $\int_0^a (a^2 - x^2)^{3/2}dx$

Set $x = a\sin\theta$. Then $x = 0$, a correspond to $\theta = 0$, $\pi/2$; when θ increases from 0 to $\pi/2$, x increases from 0 to a; and $d\sin\theta/d\theta = \cos\theta$ is continuous in $(0, \pi/2)$. Hence

$$\int_0^a (a^2 - x^2)^{3/2}dx = a^4\int_0^{\pi/2}\cos^4\theta\,d\theta = \tfrac{3}{16}a^4\pi$$

EXAMPLE 2. Find $\int_0^{2a} x(2\,ax - x^2)^{1/2}\,dx = \int_0^{2a} x[a^2 - (x-a)^2]^{1/2}\,dx$.

Set $x - a = a \sin\theta$. Then x increases from 0 to $2\,a$ when θ increases from $-\pi/2$ to $\pi/2$. Hence

$$\int_0^{2a} x[a^2-(x-a)^2]^{1/2}\,dx = a^3 \int_{-\pi/2}^{\pi/2} (1 + \sin\theta)\cos^2\theta\,d\theta$$
$$= 2\,a^3 \int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{a^3\pi}{2}$$

EXAMPLE 3. Find $\int_{-1}^{3} x(x+1)^{1/2}\,dx$ by the substitution $x + 1 = t^2$.

EXAMPLE 4. Find the area between Ox and an arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. [Fig. 72]

Area $= 2\int_{\theta=0}^{\theta=\pi} y\,dx = 2\,a^2 \int_0^{\pi} (1 - \cos\theta)^2 d\theta = 3\,a^2\pi$

EXAMPLE 5. Find the area of the ellipse $x = a\cos\phi$, $y = b\sin\phi$.

EXAMPLE 6. Find the area bounded by the curve $y^2 = x^4(a^2 - x^2)$, tracing the curve.

133. The integral $\int_0^\infty f(x)dx$. Suppose that $f(x)$ is continuous in the interval (a,x) and that $\int_a^x f(x)dx$ approaches a finite limit when $x \to \infty$; we then represent that limit by the symbol $\int_a^\infty f(x)dx$. Hence, by definition,

$$\int_a^\infty f(x)dx = \lim_{x\to\infty} \int_a^x f(x)dx \tag{1}$$

We define $\int_{-\infty}^{b} f(x)dx$ in a similar manner, and $\int_{-\infty}^{\infty} f(x)dx$ by aid of the formula $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^\infty f(x)dx$. If $\int_a^x f(x)dx$ does not approach a finite limit when $x \to \infty$, $\int_a^\infty f(x)dx$ is meaningless; similarly $\int_{-\infty}^{b} f(x)dx$.

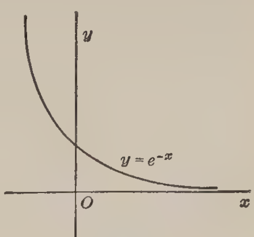

FIG. 88.

EXAMPLE 1. We have

$$\int_0^x e^{-x}dx = [-e^{-x}]_0^x = 1 - e^{-x}.$$

Hence $\int_0^\infty e^{-x}dx = \lim_{x\to\infty} (1 - e^{-x}) = 1$

Observe that $\int_0^\infty e^{-x}\,dx$ is the area between the curve $y = e^{-x}$, its asymptote $y = 0$, and Oy.

If $\phi(x)$ is a known integral of $f(x)$ in the interval (a, x), and $\phi(x)$ approaches a finite limit when $x \to \infty$, then

$$\int_a^\infty f(x)dx = \lim_{x\to\infty} \phi(x) - \phi(a) = \phi(\infty) - \phi(a) \tag{2}$$

Example 2. Show that $\int_1^\infty \frac{dx}{x^{3/2}} = 2$, but that

$$\lim_{x\to\infty}\int_1^x \frac{dx}{x^{1/2}} = \infty \text{ and } \lim_{x\to\infty}\int_1^x \frac{dx}{x} = \infty.$$

Example 3. Show that $\int_a^\infty \frac{dx}{(x-c)^n}$, $a > c$, exists when and only when $n > 1$.

Example 4. Trace the curve $y = 1/(x^2+1)$ and show that the area between the curve and its asymptote $y = 0$ is $\int_{-\infty}^\infty \frac{dx}{x^2+1} = \pi$.

Example 5. Show that $\int_3^\infty \frac{dx}{x^2 - 2x} = \log\sqrt{3}$. What area does this represent?

134. The integral $\int_a^b f(x)dx$ when $f(x)$ is ∞ in (a, b). Suppose that $f(x)$ is continuous in (a, b) except at b, but that $f(b) = \infty$. If $\int_a^x f(x)dx$ approaches a finite limit when $x \to b$ we represent that limit by $\int_a^b f(x)dx$ and write

$$\int_a^b f(x)dx = \lim_{x\to b}\int_a^x f(x)dx \tag{1}$$

Similarly, if $f(a) = \infty$, we have $\int_a^b f(x)dx = \lim_{x\to a}\int_x^b f(x)dx$.

If $f(x)$ is ∞ at some point c between a and b, but continuous elsewhere in (a, b), we define $\int_a^b f(x)dx$, when it exists, by the formula

$$\int_a^b f(x)dx = \lim_{x\to c}\int_a^x f(x)dx + \lim_{x\to c}\int_x^b f(x)dx \tag{2}$$

Example 1. Prove that $\int_0^1 \frac{dx}{(1-x^2)^{1/2}} = \frac{\pi}{2}$.

Here $f(x) = \frac{1}{(1-x^2)^{1/2}}$ is ∞ at $x = 1$; but $\int_0^x \frac{dx}{(1-x^2)^{1/2}} = \sin^{-1}x$, and $\lim_{x\to 1}\sin^{-1}x = \frac{\pi}{2}$.

If $\phi(x)$ be a known integral of $f(x)$, *and if $\phi(x)$ be continuous in (a, b)*, then, even when $f(x)$ is ∞ at one or more points of (a, b), the integral $\int_a^b f(x)dx$ exists and is given by the formula

$$\int_a^b f(x)dx = \phi(b) - \phi(a) \tag{3}$$

Thus let $f(x)$ be ∞ at $x = c$ in (a, b). Since $\phi(x)$ is continuous at $x = c$, $\lim_{x \to c} \phi(x) = \lim_{x \to c} \phi(x) = \phi(c)$. Hence, by (2),

$$\int_a^b f(x)dx = [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] = \phi(b) - \phi(a)$$

But when $\phi(x)$ becomes ∞ in (a, b), then (3) is false.

Thus $\int_{-1}^{1} \frac{dx}{x^{2/3}} = [3\,x^{1/3}]_{-1}^{1} = 6$ is true though $1/x^{2/3}$ is ∞ at $x = 0$; for $3\,x^{1/3}$ is continuous in $(-1, 1)$. But $\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -2$ is false, $-\frac{1}{x}$ being ∞ at $x = 0$; in fact, by (2), $\int_{-1}^{1} \frac{dx}{x^2} = \infty$.

Example 2. Show that $\int_1^2 \frac{dx}{(2 - x)^{3/4}} = 4$, but that

$$\lim_{x \to 2} \int_1^x \frac{dx}{2 - x} = \infty, \lim_{x \to 2} \int_1^x \frac{dx}{(2 - x)^{4/3}} = \infty.$$

Example 3. Show that $\int_a^b \frac{dx}{(b - x)^n}$ and $\int_a^b \frac{dx}{(x - a)^n}$ exist when and only when $n < 1$.

Example 4. Show that $\int_a^b \frac{dx}{(x - c)^{p/q}}$, $a < c < b$, exists when and only when q is odd and $p/q < 1$.

Example 5. Draw the curve $y^2 = 1/x$ and show that the area bounded by the curve, its asymptote $x = 0$ and the line $x = 1$ is 4.

Example 6. Show that the area bounded by a branch of the hyperbola $xy = 1$ and its asymptotes $x = 0$ and $y = 0$ is infinite.

EXERCISE XXV

Verify the following, 1 – 11.

1. $\int_{-1}^{1} \frac{dx}{x^{4/5}} = 10$

2. $\int_0^\infty e^{-3x}dx = \frac{1}{3}$

3. $\int_0^\infty \frac{dx}{x^2 + a^2} = \frac{\pi}{2\,a}$

4. $\int_{-a}^{a} \frac{dx}{\sqrt{a^2 - x^2}} = \pi$

5. $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 4\,x + 5} = \pi$

6. $\int_1^3 \frac{dx}{\sqrt{(x - 1)(3 - x)}} = \pi$

7. $\int_1^\infty \frac{dx}{x\sqrt{2\,x^2 - 1}} = \frac{\pi}{4}$

8. $\int_2^\infty \frac{dx}{x^2 - 1} = \log 3^{1/2}$

9. $\int_a^b \frac{dx}{\sqrt{(x - a)(b - x)}} = \pi$

10. $\displaystyle\int_1^2 \frac{dx}{\sqrt{x^2-1}} = \log(2+\sqrt{3})$

11. $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^n} = \frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{1}{2}\cdot\frac{\pi}{2}$ [let $x = \tan\theta$].

Find the values of the following, 12–16.

12. $\int_{-\pi/2}^{\pi/2} \cos^4 x \sin^3 x\, dx$

13. $\int_0^\pi (\sin x + \cos x)^3\, dx$

14. $\int_0^{2\pi} \sin^2 x \cos^2 x\, dx$

15. $\int_0^{\pi/2} x \cos x\, dx$

16. $\displaystyle\int_0^1 \frac{x^3\, dx}{\sqrt{1-x^2}}$ [substitute $x = \sin\theta$].

17. Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ by aid of the substitution $x = a - t$.

18. Show that $\int_0^{\pi/2} \sin^{n+1} x\, dx < \int_0^{\pi/2} \sin^n x\, dx$ and $\int_0^{\pi/4} \tan^{n+1} x\, dx < \int_0^{\pi/4} \tan^n x\, dx$.

19. Find the area between the "witch" $y = 8a^3/(x^2+4a^2)$ and its asymptote.

Find the areas bounded by the following curves and lines, 20–25.

20. $xy^2 = (x-1)^2,\ x = 0$

21. $y^2x(1-x) = 1,\ x = 0,\ x = 1$

22. $x(y-x)^2 = 1,\ x = 0,\ x = 1$

23. $(x+1)y^2 = 2-x,\ x = -1$

24. $(a^2-x^2)y^2 = x^4,\ x = a,\ x = -a.$

25. $y^2(2a-x) = x^3,\ x = 2a.$

26. Prove the following theorems, where $a > 0$, $n > 1$, and l denotes a given positive number:

(1) $\displaystyle\int_a^x \frac{dx}{x^n}$ increases as x increases and $\to 1/(n-1)a^{n-1}$ as $x \to \infty$.

(2) If $f(x)$ is positive (or 0) and $< l/x^n$ in (a, x), then $\int_a^x f(x)dx$ increases with x, but remains $< l/(n-1)a^{n-1}$; it therefore approaches a limit, and $\int_a^\infty f(x)dx$ exists.

(3) The integral $\int_a^\infty f(x)dx$ also exists when $|f(x)| < l/x^n$; for $\int_a^x f(x)dx = \int_a^x \{f(x)+|f(x)|\}\,dx - \int_a^x |f(x)|\,dx$, and these integrals are of the type (2), $f(x)+|f(x)|$ being positive (or 0) and $< 2l/x^n$.

(4) The integral $\int_a^\infty f(x)dx$ exists if $f(x)$ is continuous in (a, x) and an exponent $n > 1$ can be found such that $x^n f(x)$ approaches a finite limit when $x \to \infty$.

27. Prove that $\int_a^b f(x)dx$ exists when $f(x)$ is ∞ at $x = a$ but is continuous elsewhere in (a, b), in case an exponent $n < 1$ can be found such that $(x-a)^n f(x)$ approaches a finite limit when $x \to a$.

28. Prove the existence of the following integrals:

(1) $\displaystyle\int_0^{\infty}\frac{x-1}{x^4+3x+3}\,dx$ (2) $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{x^4+3}\,dx$

(3) $\int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad m,\ n>0$

29. Show by integration by parts and § 101 that, when n is a positive integer, $\int_0^{\infty} x^n e^{-x}dx = n!$

135. General definition of area. The definition of area in § 121 is a particular case of the following more general definition.

Let S be a piece of the plane bounded by curve arcs or by curve arcs and line segments. Let E_n and I_n be the areas of two varying rectilinear figures of n sides, the first containing S, the second contained by S, and such that, as $n \to \infty$, E_n and I_n approach the same limit l. We call l *the area of* S. The value of l is independent of the choice of the figures containing and contained by S. They may be any figures, rectilinear or not, whose areas E_n, I_n are known.[1]

136. Circular sectors. The area of a circular sector AOB of radius a and angle θ is $a^2\theta/2$. For divide AOB into n equal parts and in and about each part AOP construct triangles AOP and $A'OP'$ as in Fig. 89. Then, as $n \to \infty$,

$$\Sigma AOP/\Sigma A'OP' = OQ^2/OQ'^2 \to 1$$

and

$$\Sigma AOP = \frac{OQ}{2}\Sigma AP \to \frac{a}{2}\text{arc }AB = \frac{a^2\theta}{2}$$

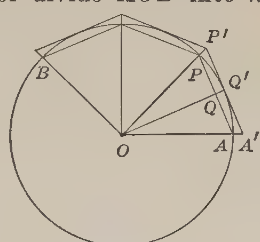

FIG. 89.

[1] For let E'_m, I'_m be the areas of any second pair of varying figures, the one containing and the other contained by S, and such that E'_m, I'_m approach the same limit l'. Then $l' = l$. For if $l' \neq l$, suppose $l > l'$ and set $l - l' = d$. We then have

$E_n - I'_m > d$ (since $E_n > l$ and $I'_m < l'$), also $E'_m - I_n > 0$ (since $E'_m > I_n$)

and therefore $\qquad (E_n - I_n) + (E'_m - I'_m) > d$

which is impossible since $E_n - I_n \to 0$ and $E'_m - I'_m \to 0$. Hence $l' = l$.

137. Areas, polar coordinates. Let AOB be the space bounded by a continuous curve arc AB whose equation in polar coordinates is $r = f(\theta)$, and the rays $\theta = \alpha$ and $\theta = \beta$ (OA and OB). Divide AOB into n parts whose angles $\delta\theta$ all $\rightarrow 0$ when $n \rightarrow \infty$. In and about each part POQ construct circular sectors POR and QOT whose radii are the least and greatest values of r in that part. The space AOB contains the sum of the inner sectors and is contained by the sum of the outer sectors. The areas of both sector sums are sums of the form $\Sigma r^2\delta\theta/2$ in which the r of each term is a value of $r = f(\theta)$ in the $\delta\theta$ of that term. Hence both areas $\rightarrow \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$ when $n \rightarrow \infty$, § 124. Therefore the area of AOB is

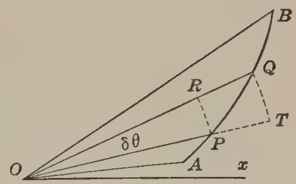

FIG. 90.

$$A_\alpha^\beta = \tfrac{1}{2}\int_\alpha^\beta r^2 d\theta \tag{1}$$

EXAMPLE. Find the area bounded by the cardioid $r = a(1 - \cos\theta)$ (Fig. 45, p. 78).

Since $\cos\theta = \cos(-\theta)$, the curve is symmetric to Ox. Hence

$$\text{Area} = 2\frac{a^2}{2}\int_0^\pi (1 - \cos\theta)^2 d\theta = a^2\int_0^\pi (1 - 2\cos\theta + \cos^2\theta)d\theta = \frac{3\,a^2\pi}{2}$$

EXERCISE XXVI

Find the areas bounded by the following curves and lines, drawing the graphs.

1. $r = a(1 - \cos\theta)$, $\theta = \pi/6$, $\theta = \pi/3$

2. $r = a\cos\theta$, $\theta = 0$, $\theta = \pi/4$

3. $r = a\tan\theta$, $\theta = 0$, $\theta = \pi/6$

4. $r = a\theta$, $\theta = 0$, $\theta = 2\pi$

5. $r = e^{2\theta}$, $\theta = 0$, $\theta = \pi/2$

6. $r^2 = a^2\cos 2\theta$ [Fig. 47]

7. A loop of $r = a\cos 3\theta$ [Fig. 46]

8. A loop of $r = a\cos 2\theta$

9. The parabola $r = a\sec^2\dfrac{\theta}{2}$, $\theta = 0$, $\theta = \dfrac{\pi}{2}$

10. $r = 2 - \cos\theta$

11. $r = \cos\theta$ and $r = \sin\theta$

12. The loop of $r = \sin^3\theta/3$.

VOLUMES OF SOLIDS

138. Definition of volume. The notion of volume may be extended from solids for which it is already defined (right prisms) to other solids by the following definition (see § 135).

Let S be a given solid and let E_n, I_n be the volumes of two varying solids of known volume, one containing S, the other contained by S, and such that, as $n \to \infty$, E_n and I_n approach the same limit l. We call l the *volume of* S.

The value of l is independent of the choice of the varying solids (see footnote p. 153).

139. Right cylinders. The volume of a right cylinder K of altitude a and base area B is aB. For in and about the base of K we can construct polygons whose areas approach B as limit. The right prisms of altitude a on these polygons include K between them, and their volumes both approach aB as limit.

1. The volume of a *right circular cylinder* of altitude a and radius r is $ar^2\pi$.

2. The area of a circular ring of inner radius r_1 and outer radius r_2 is $\pi(r_2^2 - r_1^2) = \pi(r_2 + r_1)(r_2 - r_1)$. Hence for the volume V of the *cylindrical shell* of altitude a on this ring, we have

$$V = a\,2\,\pi r\,\delta r, \text{ where } r = (r_1 + r_2)/2 \text{ and } \delta r = r_2 - r_1$$

140. Solids of revolution. 1. Let S_x denote the solid got by revolving the space $abBA$ about Ox, AB being a continuous arc of a curve $y = f(x)$.

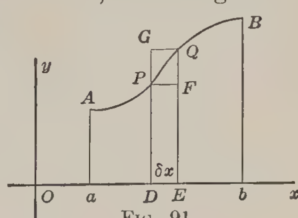

FIG. 91.

By parallels to Oy, divide $abBA$ into n strips $PDEQ$ on parts δx of ab which $\to 0$ when $n \to \infty$, and in and about each strip construct rectangles PE and QD whose altitudes are the least and

greatest curve ordinates in the strip. The part of S_x generated by $PDEQ$ is included between the cylinders generated by PE and QD, and the volumes of these cylinders are $DP^2\pi\,\delta x$ and $EQ^2\pi\,\delta x$. Hence S_x is included between two cylinder sums whose volumes are sums $\pi\Sigma y^2\,\delta x$ in both of which the y of each term is a value of $y = f(x)$ in the δx of that term, and which, therefore, $\rightarrow \pi\int_a^b y^2\,dx$ when $n \rightarrow \infty$. Hence the volume V_x of S_x is

$$V_x = \pi\int_a^b y^2\,dx \tag{1}$$

2. It being supposed that the arc AB does not cut Oy, let S_y be the solid got by revolving $abBA$ about Oy. The part of S_y generated by $PDEQ$ is included between the cylindrical shells generated by PE and QD, and by § 139, 2, the volumes of these shells are $2\,\pi x'DP\,\delta x$ and $2\,\pi x'EQ\,\delta x$ where x' is the midpoint of δx. Both of these products are of the form $2\,\pi x'f(x)\delta x$ where x and x' are points of δx, and by a theorem to be proved later, § 145, $\Sigma x'f(x)\delta x \rightarrow \int_a^b xf(x)dx$. Hence the volume V_y of S_y is

$$V_y = 2\,\pi\int_a^b xy\,dx \tag{2}$$

3. Similarly the volumes of the solids got by revolving about Oy and Ox the space bounded by the curve $x = \phi(y)$ and the lines $x = 0$, $y = c$, $y = d$ is

$$V_y = \pi\int_c^d x^2\,dy \quad (3) \qquad V_x = 2\,\pi\int_c^d xy\,dy \tag{4}$$

Example 1. For the sphere of radius a got by revolving the circle $x^2 + y^2 = a^2$ about Ox,

$$V_x = \pi\int_{-a}^{a} y^2\,dx = 2\,\pi\int_0^a (a^2 - x^2)dx = \tfrac{4}{3}\,a^3\pi.$$

Example 2. Find the volumes of the solids got by revolving about Ox and Oy the space bounded by the lines $y_1 = x$, $y_2 = 2\,x$, $x = 1$, $x = 3$.

$$V_x = \pi\int_1^3 (y_2^2 - y_1^2)dx = \pi\int_1^3 3\,x^2\,dx = 26\,\pi$$

$$V_y = 2\,\pi\int_1^3 x(y_2 - y_1)dx = 2\,\pi\int_1^3 x^2\,dx = \tfrac{52}{3}\,\pi$$

EXERCISE XXVII

1. Find by integration the volumes of the cones got by revolving about Ox and Oy the triangle bounded by Ox, Oy, and the line $3x + 2y = 6$.

2. Prove that the volume of a right circular cone of altitude a and base radius r is $ar^2\pi/3$.

3. Prove that the volumes of the *prolate spheroid* and the *oblate spheroid* got by revolving the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a > b$, about its major and minor axes are $\frac{4}{3}ab^2\pi$ and $\frac{4}{3}a^2b\pi$.

Find the volumes got by revolving about Ox the spaces bounded by the following curves:

4. $y^2 = x^3$, $x = 4$

5. $y^2 = x^3$, $y = x$

6. $y^2 = x^3$, $y = 4$, $y = 2x$

7. $y^2 = 4x$, $2y = x$

8. $y^2 = 4x$, $x^2 + y^2 = 5x$

9. $y = x^2$, $y = x$, $y = 2x$

10. An arch of $y = \sin x$

11. $y^2 = 1/(x^2 + 1)$, $y = 0$

12. $x^2 + y^2 = 9$, $x = 2$, $x = 3$

13. $y = 8a^3/(x^2 + 4a^2)$, $y = 0$

14. $x = a\cos\theta$, $y = a\sin\theta$, $\theta = 0$ and $\theta = \pi/4$

15. The loop of $y^2 = x^2(3 - x)$

16. An arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

Find the volumes got by revolving about Oy the spaces bounded by the following curves:

17. $x^2 + y - 2 = 0$, $y = 0$

18. $y^2 = 4x$, $x = 0$, $y = 4$

19. $y = x^2$, $y = x$, $y = 2x$

20. $xy = 1$, $x = 0$, $x = 1$

21. $x^2 - y^2 = a^2$, $y = -a$, $y = a$

22. $x^2 + 3y^2 - 6y = 0$

23. $y = e^x$, $x = 0$, $y = 4$

24. Segment of $x^2 + y^2 + 2x = 3$ to the right of Oy

25. $y = x$, $y = x + 2$, $x = 2$, $x = 4$

26. $3x - 2y = 1$, $2x + y = 3$, $5x - y = 11$

27. $y^2/a^2 = (a - x)/x$, $x = 0$

28. $y^2 = (2 - x)^3/x$, $x = 0$

29. Show that the volume of the solid got by revolving about the line $x = 2$ the parabolic segment between $y^2 = 4x$ and $x = 2$ is $4\pi\int_0^2 y(2 - x)dx$, and find its value.

30. The space bounded by the circle $x^2 + y^2 = a^2$ and its tangents $x = a$ and $y = a$ is revolved about $x = a$. Find the volume of the solid thus generated.

31. Prove that the volume of the *torus* [Fig. 98] got by revolving the circle $x^2 + (y - b)^2 = a^2$, $a < b$, about Ox is $2\,ba^2\pi^2$.

32. The axis of a cylinder of radius $4\,a$ passes through the center of a sphere of radius $5\,a$; show that the volume of the part of the sphere exterior to the cylinder is $36\,\pi a^3$.

33. Find a formula for the volume of a spherical segment.

141. Other solids. Let A_x denote the section of a solid K by a plane perpendicular to Ox at the distance x from O. The area of A_x is a function of x; represent it by $A(x)$, and suppose that $A(x)$, is continuous between $x = a$ and $x = b$. Divide the segment ab of Ox into n parts δx which $\to 0$ when $n \to \infty$. The sections A_x through the end points of any part δx cut out a part ΔK of K; suppose that the right cylinders of altitude δx on these sections include ΔK between them; then since the volumes of both cylinders are of the form $A(x)\delta x$, where x is a point in δx, we may conclude, as in § 140, that the volume V_a^b of K between $x = a$ and $x = b$ is

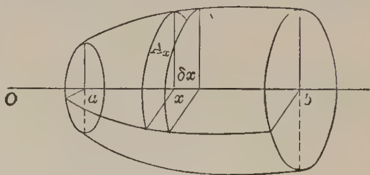

FIG. 92.

$$V_a^b = \int_a^b A(x)\,dx \tag{1}$$

EXAMPLE 1. The volume of the *ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is $(4/3)\,abc\pi$.

For the plane $x = x'$ perpendicular to Ox at the distance x' from O cuts the surface in an ellipse whose equation in the plane $x = x'$ is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x'^2}{a^2}$$

The semiaxes of this ellipse are $b\left(1 - \frac{x'^2}{a^2}\right)^{1/2}$, $c\left(1 - \frac{x'^2}{a^2}\right)^{1/2}$; hence its area is $bc\left(1 - \frac{x'^2}{a^2}\right)\pi$, p. 145, Ex. 24.

Therefore

$$V = 2\int_0^a bc\left(1 - \frac{x^2}{a^2}\right)\pi\,dx = \frac{4}{3}\,abc\pi$$

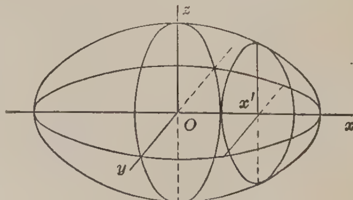

FIG. 93.

EXAMPLE 2. Find the volume of the piece cut from the part of a right cylinder standing on the quadrant OAB of radius a by the plane $OAQC$, the angle BOC being 30°.

Every section of $OABC$ by a plane perpendicular to Oy is a triangle PDQ similar to OBC, and if $OP = y$, then $PD = (a^2 - y^2)^{1/2}$, $DQ = PD \tan 30$,

$$\therefore PDQ = \frac{\sqrt{3}}{6}(a^2 - y^2)$$

Hence $V = \dfrac{\sqrt{3}}{6}\int_0^a (a^2 - y^2)\,dy = \dfrac{a^3}{3\sqrt{3}}$

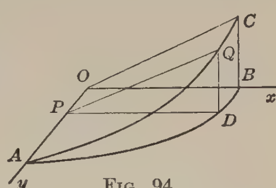

FIG. 94.

EXAMPLE 3. Find the volume bounded by the elliptic paraboloid $4x = y^2 + 2z^2$ and the plane $x = 5$.

EXAMPLE 4. Solve Ex. 2, using sections perpendicular to Ox.

EXAMPLE 5. Let BC be the major axis of the ellipse $x^2 + 2y^2 = 2$, PQ any chord perpendicular to BC, and A_x the equilateral triangle on PQ and perpendicular to BC. What is the volume of the solid generated when A_x moves from B to C?

EXAMPLE 6. In the pyramid O–DEF, the angles ODE, ODF are right angles. Show that the volume is $1/3$ the altitude times the area of the base. Hence show that the volume of any pyramid or cone of altitude a and base area B is $(1/3)aB$.

CURVE ARCS AND SURFACES OF REVOLUTION

142. Length of a curve arc. In an arc AB of a curve $y = f(x)$ along which $f'(x)$ is continuous, inscribe a polygon of n sides all of which $\to 0$ when $n \to \infty$.

Let PQ be any side of this polygon and δx its projection on Ox. By § 97 (7), there is a point on the arc PQ where the slope angle equals DPQ; let x denote the abscissa of this point. Then since $\tan DPQ = f'(x)$, we have

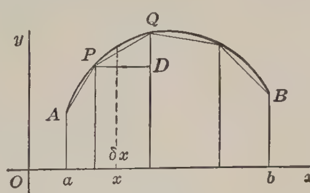

FIG. 95.

$$PQ = PD \sec DPQ = \delta x\,[1 + f'(x)^2]^{1/2}$$

Hence the perimeter ΣPQ of the polygon can be expressed in the form $[\Sigma(1 + f'(x)^2)^{1/2}\delta x]_a^b$; it therefore approaches a definite limit when $n \to \infty$, § 124. We call this limit, namely $\int_a^b [1 + f'(x)^2]^{1/2}dx$, the *length of the arc AB*; and representing it by s_a^b, have the formula

$$s_a^b = \int_a^b \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx \tag{1}$$

EXAMPLE 1. Find the length of the arc of $y = x^{3/2}$ between $x = 0$ and $x = 4/3$.

$$s = \int_0^{4/3} (1 + \tfrac{9}{4}x)^{1/2}\, dx = \tfrac{8}{27}[(1 + \tfrac{9}{4}x)^{3/2}]_0^{4/3} = \tfrac{8}{27}(8 - 1) = \tfrac{56}{27}.$$

Let s denote the length of the arc of $y = f(x)$ between any fixed point $P_0(x_0, y_0)$ on the curve and the variable point $P(x, y)$, $f'(x)$ being continuous along P_0P. Then

$$s = \int_{x_0}^{x} \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx, \quad \frac{ds}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} \tag{2}$$

Multiplying both sides of (2) by dx, we obtain

$$ds = [(dx)^2 + (dy)^2]^{1/2} \tag{3}$$

When the curve is given by parametric equations $x = \phi(t)$, $y = \psi(t)$, we have $dx = \phi'(t)dt$, $dy = \psi'(t)dt$, and therefore by (3),

$$ds = [\phi'(t)^2 + \psi'(t)^2]^{1/2}dt \tag{4}$$

When the curve is given by a polar equation $r = \phi(\theta)$, referred to the same origin and Ox as in Fig. 95, we have $x = r\cos\theta$, $y = r\sin\theta$, where $r = \phi(\theta)$; hence

$$\frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta \qquad \frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$$

In (4) replace dt, $\phi'(t)$, $\psi'(t)$ by $d\theta$, $dx/d\theta$, $dy/d\theta$; we obtain

$$ds = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{1/2} d\theta \tag{5}$$

We have identically,

$$s = \int_{P_0}^{P} ds \tag{6}$$

In (6) we may replace ds by its expression in terms of any variable if we also replace P_0, P by the appropriate limits. Hence, if t_0, t and θ_0, θ correspond to P_0, P, we have from (4) and (5)

$$s = \int_{t_0}^{t} [\phi'(t)^2 + \psi'(t)^2]^{1/2}\,dt \quad (7) \qquad s = \int_{\theta_0}^{\theta} \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{1/2} d\theta \quad (8)$$

EXAMPLE 2. Find the length of the hypocycloid of four cusps $x = a\cos^3\theta$, $y = a\sin^3\theta$.

Substituting in (7), we find for the length s of the first of the four equal quadrants of the curve

$$s = 3\,a\int_0^{\pi/2} [\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta]^{1/2}\,d\theta$$
$$= 3\,a\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = \frac{3\,a}{2}.$$

EXAMPLE 3. Find the length of the cardioid $r = a(1 - \cos\theta)$. Substituting in (8), we find for the length s of half the curve

$$s = a\int_0^{\pi} [(1 - \cos\theta)^2 + \sin^2\theta]^{1/2}\,d\theta = a\int_0^{\pi}[2 - 2\cos\theta]^{1/2}\,d\theta$$
$$= 2\,a\int_0^{\pi} \sin\frac{\theta}{2}\,d\theta = 4\,a.$$

EXERCISE XXVIII

Find the following curve lengths, drawing the graphs.

1. $9\,y^2 = 4\,x^3$, $x = 3$ to $x = 8$
2. $y = \log\cos x$, $x = \frac{1}{2}$ to $x = \frac{3}{2}$
3. $x^2 + y^2 = 4$, $x = 1$ to $x = 2$
4. Arc of $y^2 = 6\,x - x^2$ intercepted by $y = x$
5. $y = \log(1 - x^2)$, $x = -\frac{1}{2}$ to $x = \frac{1}{2}$
6. $y = 2\sec x$, $x = 0$ to $x = \pi/4$
7. $y = ax^2$, $x = 0$ to $x = x$
8. $y^2 = 16\,x$, $x = 0$ to $x = 9$
9. $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$, $x = 0$ to $x = a$
10. Hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$
11. $y = \log x$, $x = 1$ to $x = 2$
12. $9\,y^2 = 4(x^2 - 1)^3$, $x = 1$ to $x = 3$
13. Loop of $9\,ay^2 = x^2(x + 3\,a)$
14. Loop of $y^2 = x^2(\frac{1}{3} - x)$
15. $r = e^{\theta}$, $\theta = 0$ to $\theta = 2\,\pi$
16. $r = a\cos\theta$, $\theta = 0$ to $\pi/4$
17. $r = a\cos^2\frac{\theta}{2}$, $\theta = 0$ to $\theta = \pi$
18. $r = a\cos\theta + b\sin\theta$

19. $x = a(\sin\theta - \theta\cos\theta)$, $y = a(\cos\theta + \theta\sin\theta)$, $\theta = 0$ to $\theta = \pi$

20. An arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

21. Prove respecting curves of the type $x = \phi(t)$, $y = \psi(t)$ that:

If $$\phi'(t) = f(t)(at + b),\ \psi'(t) = f(t)(a't + b'),$$

then $$s_0^t = \int_0^t f(t)(At^2 + Bt + C)^{1/2}\,dt$$

Hence show that an arc of $x = t^n$, $y = t^{n+1}$, or $y^n = x^{n+1}$, can be found by the formulas of integration.

143. Area of frustum of cone. In the bases of a frustum of a right circular cone inscribe similarly placed regular polygons of n sides, and then form trapezoids, like $PQP'Q'$, by joining their corresponding angular points. The limit which the sum of the areas of these trapezoids approaches when $n \to \infty$ is called the area of the lateral surface of the frustum. It is $2\pi rh$, where h is the slant height of the frustum and r is the radius of the midsection parallel to the bases.

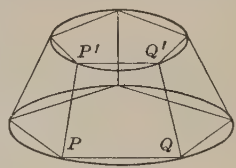

Fig. 96.

144. Area of a surface of revolution. When the arc AB of $y = f(x)$ considered in § 142 is revolved about Ox it generates a surface. The area of this surface is defined as the limit of the area generated by the inscribed polygon of n sides described in § 142, when $n \to \infty$. Each side PQ of the polygon generates the lateral surface of the frustum of a right circular cone of which PQ is the slant height; its area is $2\pi \cdot EM \cdot PQ$, where M is the midpoint of PQ. As shown in § 142, there is a point x in δx, the projection of PQ on Ox, such that $PQ = [1 + f'(x)^2]^{1/2}\,\delta x$. And since EM is between the least and greatest values of $f(x)$ in δx, there is also a point

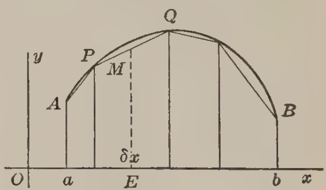

Fig. 97.

x' in δx such that $EM = f(x')$. Hence the area generated by the polygon can be expressed in the form

$$\Sigma 2\,\pi \cdot EM \cdot PQ = 2\,\pi \Sigma f(x')[1 + f'(x)^2]^{1/2}\,\delta x$$

By § 145, $\Sigma f(x')[1 + f'(x)^2]^{1/2}\,\delta x \to \int_a^b f(x)[1 + f'(x)^2]^{1/2}\,dx$. Hence

$$\text{Area generated by arc } AB = 2\,\pi \int_{x=a}^{x=b} y\,ds \qquad (1)$$

EXAMPLE 1. Find the area of the surface got by revolving either arc of the parabola $y^2 = 4\,x$ between $x = 0$ and $x = 8$ about Ox.

$$S = 2\,\pi \int_0^8 2\,x^{1/2}\left[1 + \frac{1}{x}\right]^{1/2} dx = 4\,\pi \int_0^8 (x+1)^{1/2}\,dx = \tfrac{8}{3}\,\pi(27 - 1) = \tfrac{208}{3}\,\pi.$$

EXAMPLE 2. If an arc of the circle $x^2 + y^2 = a^2$ between $x = x_1$ and $x = x_1 + h$ be revolved about Ox, a *spherical zone* Z of altitude h is generated. Prove that area $Z = 2\,\pi a h$.

$$x^2 + y^2 = a^2 \text{ gives } x + y\frac{dy}{dx} = 0 \qquad \therefore 1 + \left(\frac{dy}{dx}\right)^2 = \frac{x^2 + y^2}{y^2} = \frac{a^2}{y^2}.$$

$$\text{Hence area } Z = 2\,\pi \int_{x_1}^{x_1+h} \frac{a}{y}\,y\,dx = 2\,\pi a h.$$

Therefore the area of the sphere is $2\,\pi a \cdot 2\,a = 4\,\pi a^2$.

In (1) we may express ds and y in terms of any variable. If the arc is circular, it is usually most convenient to make the angle θ at the center the independent variable, which gives $ds = a d\theta$, a being the radius.

EXAMPLE 3. Find the area of the torus got by revolving $x^2 + (y - b)^2 = a^2$, $a < b$, about Ox. Since $y = DP = OC - EC = b - a \cos\theta$, and $ds = a d\theta$, we have $(1/2)S = 2\,\pi \int_0^\pi (b - a\cos\theta) a d\theta = 2\,ab\pi^2$, $S = 4\,ab\pi^2$. By the same method we can find the areas of the surfaces generated by the arcs into which Ox cuts the circle when $b < a$.

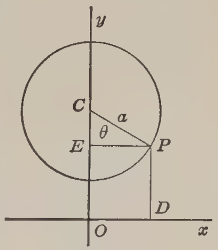

FIG. 98.

145. Theorem. *The functions $f(x)$ and $\phi(x)$ are continuous in (a, b). Divide (a, b) into n parts δx, all of which $\to 0$ when $n \to \infty$, and let x and x' denote any two points in the same δx; then*

$$\lim\,[\Sigma f(x)\phi(x')\delta x]_a^b = \int_a^b f(x)\phi(x)dx \qquad (1)$$

For we can express $\Sigma f(x)\phi(x')\delta x$ in the form

$$\Sigma f(x)\phi(x)\delta x + \Sigma f(x)[\phi(x') - \phi(x)]\delta x$$

The limit of the first term is $\int_a^b f(x)\phi(x)dx$; hence the theorem will be proved if it can be shown that the limit of the second term is 0.

But, since $\phi(x)$ is continuous, if any positive number ϵ be assigned we can find a value n' of n such that when $n > n'$ each of the differences $\phi(x') - \phi(x)$ is numerically less than ϵ (§ 122). Hence, if M denotes the greatest value of $|f(x)|$ in (a, b) we shall have, when $n > n'$,

$$|\Sigma f(x)[\phi(x') - \phi(x)]\delta x| < M\epsilon\Sigma\delta x = M\epsilon(b - a)$$

We can take $M\epsilon(b - a)$ as small as we please; hence

$$\lim \Sigma f(x)[\phi(x') - \phi(x)]\delta x = 0$$ [1]

EXERCISE XXIX

Find the areas got by revolving the following arcs about Ox.

1. $y = x^3$, $x = 0$ to $x = 1$ **2.** $y = 2\sec x$, $x = 0$ to $x = \pi/4$

3. Loop of $9y^2 = x(x - 3)^2$ **4.** Loop of $8y^2 = x^2 - x^4$

5. $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$, $x = 0$ to $x = a$

6. $y^2 = 1 - x$, $x = 0$ to $x = 1$

7. $x^2 + 2y^2 = 2$ **8.** $x^2 - y^2 = 1$, $x = 1$ to $x = 2$

9. $x = a\cos\theta$, $y = a\sin\theta$, $\theta = \pi/4$ to $\theta = \pi/3$.

10. An arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

11. The cardioid $r = a(1 - \cos\theta)$ **12.** $y = e^{-x}$, $x = 0$ to $x = \infty$

13. $x = 1/(t + 1)$, $y = t/(t + 1)$, $t = 0$ to $t = 1$

14. $x = 1/(t^2 + 1)$, $y = t/(t^2 + 1)$, $t = 0$ to $t = 5$

15. $6xy = x^4 + 3$, $x = 2$ to $x = 4$ **16.** $x = a\cos^3\theta$, $y = a\sin^3\theta$

[1] Similarly, if $f(x)$, $\phi(x)$, $\psi(x)$ are continuous in (a, b), and x, x', x'' denote points in the same δx, we have

$$\lim\,[\Sigma f(x)\phi(x')\psi(x'')\delta x]_a^b = \int_a^b f(x)\phi(x)\psi(x)dx$$

This follows from § 145 and the fact that

$$f(x)\phi(x')\psi(x'') = f(x)\phi(x)\psi(x'') + f(x)\psi(x'')[\phi(x') - \phi(x)].$$

The extension to the case of any number of factors is obvious.

17. Find the area of a parabolic mirror 2 ft. deep and 6 ft. wide.

18. Show that the area S got by revolving about Oy the arc of $y^2 = 4x$ to the left of the line $x - 1 = 0$ is given by $S/2 = 2\pi\int_{x=0}^{x=1} x\,ds$ and find it.

19. Find the area got by revolving the arc in Ex. 18 about $x - 1 = 0$.

20. An arc of length λ is revolved about each of two parallel axes whose distance apart is d; show that the difference between the areas generated is $2\pi d\lambda$.

21. An arc which subtends the angle 2β at the center of a circle of radius a is revolved about its chord; show that the area generated is $4\pi a^2(\sin\beta - \beta\cos\beta)$.

22. Show that the area generated by revolving a quadrant of a circle of radius a about the tangent at one extremity is $\pi(\pi - 2)a^2$.

23. A variable sphere of radius r has its center on the surface of a fixed sphere of radius a; show that the area of the portion of it within the fixed sphere is greatest when $r = (4/3)a$.

24. A point P moves from A to B on a curve $y = f(x)$ in the xy-plane; and Q is a point at the distance $z = \phi(x)$ vertically above P. Show that area generated by PQ is $\int_A^B z\,ds$.

25. Show that the area of the portion of the cylindrical surface $x^2 + y^2 - 2ax = 0$ within the sphere $x^2 + y^2 + z^2 = 4a^2$ is $16a^2$.

PHYSICAL PROBLEMS

146. Fluid pressure. At the distance x below the surface E of a fluid the intensity of the pressure in every direction is kx, where k denotes the weight of a cubic unit of the fluid. Let there be immersed in the fluid a vertical plane surface S whose width at the distance x below E is $w = f(x)$, $f(x)$ being continuous. By planes parallel to E divide the portion S_a^b of S between the levels $x = a$ and $x = b$ into n strips ΔS of height δx which $\to 0$ when $n \to \infty$. The pressure on the strip ΔS between x and $x + \delta x$ is $> kx\Delta S$ and $< k(x + \delta x)\Delta S$ and therefore is $kx'\Delta S$ where x' is some

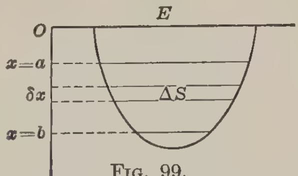

FIG. 99.

point in δx; and there is also a point x'' in δx such that $\Delta S = f(x'')\delta x$ (§ 129 (6)).

When $n \to \infty$, then $[\Sigma kx'f(x'')\delta x]_a^b \to k\int_a^b x\, f(x)dx$ (§ 145).

Therefore the pressure on S_a^b is

$$P_a^b = k\int_a^b x\, w dx \tag{1}$$

Example. A trough whose cross section is a semicircle of radius 3 ft. is full of water. What is the pressure on the end of the trough, the weight of 1 cu. ft. of water being 62 lbs.? Here $w/2 = (3^2 - x^2)^{1/2}$. Hence $P = 62\int_0^3 2\, x(9 - x^2)^{1/2}dx = 1116$ lbs.

EXERCISE XXX

1. A flood gate 10 ft. square has its top at the water surface. What is the pressure on the gate? What are the pressures on the halves into which a diagonal divides it?

2. What is the pressure on the gate closing a water main of 1 ft. radius if the center of the gate be 100 ft. below the surface of the water in the reservoir?

3. Show that the pressure on the part S_a^b of a plane surface which makes the angle α with the vertical is $k \sec \alpha \int_a^b xwdx$.

4. Show that the pressure on the part S_a^b of any surface for which the x-differential of area is of the form $dS = \phi(x)dx$ is $P_a^b = k\int_{x=a}^{x=b} x\, dS$.

5. Find the pressure on a sphere of 4 ft. radius whose center is 6 ft. below the water surface.

6. A trough 10 ft. long and whose cross section is a triangle 4 ft. deep and 6 ft. across at the top is full of water. Find the pressure on its ends and sides.

7. A cylindrical tank of length 20 ft. and radius 3 ft. is full of oil weighing 50 lbs. per cu. ft. Find the pressure on the tank when set (*a*) vertically, (*b*) horizontally.

147. Work. When, under the action of a constant force F, a particle moves the distance s in the direction of F, the force is said to do the work Fs.

Let a particle P move along Ox from $x = a$ to $x = b$ under the action of a variable force in the direction of the motion, the force being a continuous function $F(x)$ of the distance x

of P from O. Divide (a, b) into n parts δx which $\to 0$ when $n \to \infty$, and multiply each δx by the value of $F(x)$ at any point in that δx. The sum of the products thus obtained, $[\Sigma F(x)\delta x]_a^b$, is the total work done in the several parts δx by the constant forces represented by the selected values of $F(x)$. When $n \to \infty$, then $[\Sigma F(x)\delta x]_a^b \to \int_a^b F(x)dx$.

We therefore define the work done by $F(x)$ as

$$W = \int_a^b F(x)dx \tag{1}$$

If P moves in the opposite direction to that of F, the negative of (1) is called the work done against F.

EXERCISE XXXI

1. A particle P on Ox is attracted toward O by a force $F = kx^2$. What is the work done when P moves from the point $x = 6$ to the point $x = 3$?

2. A positive charge m of electricity at O repels a unit positive charge at the distance x from O with the force m/x^2. What is the work done when the unit charge is carried from $x = 2a$ to $x = a$? from $x = \infty$ to $x = a$?

3. The work done against gravity in raising a pound weight one foot is called a *foot-pound*. A cistern is bounded laterally by a cylindrical surface of depth 20 ft. and radius 6 ft. and at the bottom by an inverted hemisphere: find in ft. lbs. the work done in pumping it out if at the start it is full of water.

4. The moment about O of a weight m at the point $(x, 0)$ of Ox is mx. Show by the reasoning in § 146 that if m were distributed uniformly along the segment ab of Ox the moment would be $k\int_a^b x\,dx$, where $k = m/(b - a)$.

5. A light A of intensity k is at the distance h above the center O of a circular space S of radius a. If the intensity of the illumination of S at any point P is $\dfrac{k \sin OPA}{AP^2}$, show by the reasoning in § 146 that the total illumination of S is $k\displaystyle\int_0^a \frac{2\pi hr\,dr}{(h^2 + r^2)^{3/2}}$.

6. The equation of motion along Ox of a particle P of mass m is $x = t^3$. Find the work done by the force causing this motion between $t = 2$ and $t = 4$. $(F = md^2x/dt^2)$

XVI. DOUBLE INTEGRALS

148. Continuous functions of two variables. Let S denote a closed region, its boundary included, in the xy-plane, and let $f(x, y)$ denote a one-valued function of x and y in S, so that to each point (x, y) of S corresponds a single real value of $f(x, y)$.

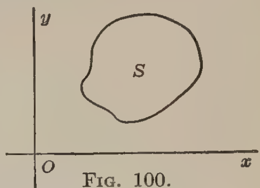

FIG. 100.

1. The function $f(x, y)$ is said to be *continuous at the point* (a, b) *of* S if $f(a, b)$ has a definite finite value and if

$$f(x, y) \to f(a, b) \text{ when } (x, y) \to (a, b) \text{ in } S$$

2. The function $f(x, y)$ is said to be *continuous in* S if it is continuous at every point of S.

3. It can be proved that among the different values of a function $f(x, y)$ continuous in S there is a *greatest value* M *and a least value* m, and that $f(x, y)$ takes in S every value between M and m.

4. The difference between the greatest and least values of $f(x, y)$ in any part of S is called the *oscillation* of $f(x, y)$ in that part.

5. It can be proved that if $f(x, y)$ is continuous in S it is *uniformly continuous in* S: that is, to any assigned positive number ϵ there corresponds another positive number δ such that in every part of S which can be inclosed in a circle of radius δ the oscillation of $f(x, y)$ is less than ϵ.

149. The integral $\int_S f(x, y)\, dS$. The proof of the following theorem is, except for notation, identical with that in § 124.

Let a region S in which $f(x, y)$ is continuous be divided into parts each of which can be inclosed in a circle of radius ρ. Let δS denote any one of these parts and also its area, and let (x, y) denote any point in δS. When $\rho \to 0$, the sum $\Sigma f(x, y)\delta S$ approaches a definite limit, and the value of this limit is independent of the mode of partition of S.

We represent $\lim \Sigma f(x, y)\,\delta S$ by the symbol $\int_S f(x, y)\,dS$, read "integral $f(x, y)$ over S." Hence, by definition,

$$\int_S f(x, y)\,dS = \lim \Sigma f(x, y)\,\delta S \tag{1}$$

150. The function $F(x) = \int_c^d f(x, y)dy$. Let $f(x, y)$ be continuous in the rectangle R bounded by the lines $x = a$, $x = b$, $y = c$, $y = d$. For any assigned value x' of x between a and b the y-integral $\int_c^d f(x', y)dy$ has a definite value; hence $\int_c^d f(x, y)dy$ is a function of x in (a, b), § 14.

The function $F(x) = \int_c^d f(x, y)dy$ is continuous.

For $f(x, y)$ is uniformly continuous in R, § 148, 5. Hence to any assigned positive number ϵ there corresponds another positive number δ such that everywhere in R

$$|f(x + \Delta x, y) - f(x, y)| < \epsilon \text{ when } |\Delta x| < \delta \tag{1}$$

and when (1) is satisfied, we have, p. 146, Ex. 6,

$$\begin{aligned} |F(x + \Delta x) - F(x)| &\leq \int_c^d |f(x + \Delta x, y) - f(x, y)|dy \\ &< \int_c^d \epsilon\, dy = \epsilon(d - c) \end{aligned} \tag{2}$$

We can take $\epsilon(d - c)$ as small as we please; hence

$$\lim_{\Delta x \to 0} F(x + \Delta x) = F(x).$$

Example. $F(x) = \int_1^3 2\,xy\,dy = [xy^2]_{y=1}^{y=3} = 8\,x$, which is a continuous function of x in any finite rectangle $x = a$, $x = b$, $y = 1$, $y = 3$.

151. The integral $\int_a^b \int_0^{\phi(x)} f(x, y)dy\,dx$. Let $f(x, y)$ be continuous in the space S bounded by the lines $x = a$, $x = b$, Ox, and a curve $y = \phi(x)$ which is continuous between $x = a$ and $x = b$ (Fig. 101).

In S the integral $\int_0^y f(x, y)dy$ is a continuous function of x (by the reasoning of § 150) and of its upper limit y (by § 126). Therefore, $y = \phi(x)$ being continuous, the integral $\int_0^{\phi(x)} f(x, y)dy$ is a continuous function of x in the interval (a, b) and has an x-integral between the limits a and b. This integral

$$\int_a^b\left[\int_0^{\phi(x)} f(x, y)dy\right]dx \text{ is written } \int_a^b\int_0^{\phi(x)} f(x, y)dy\,dx \quad (1)$$

EXAMPLE. $\int_1^3\int_0^{\sqrt{x}} 2\,y\,dy\,dx = \int_1^3 [y^2]_0^{\sqrt{x}} dx = \int_1^3 x\,dx = 4.$

152. Evaluation of the integral $\int_S f(x, y)dS$. Let S be the space $abBA$ bounded by the lines $x = a$, $x = b$, Ox, and a continuous curve arc $y = \phi(x)$ along which y increases only or decreases only as x increases. And let $\int_S f(x, y)dS$ denote the integral over this space defined in § 149. We shall prove that

$$\int_S f(x, y)dS = \int_a^b\int_0^{\phi(x)} f(x, y)dy\,dx$$

Through points $x_2, x_3, \cdots$ of Ox got by dividing (a, b) into parts, then subdividing these parts, and so on, draw parallels to Oy, and so divide S into strips $S_1, S_2, \cdots S_m$ whose bases $h_1, h_2, \cdots, h_m \to 0$ when $m \to \infty$. Then by parallels to Ox divide each strip into a set of rectangles (like QR) and a triangle (like PVT) whose altitudes $k_1, k_2, \cdots k_n \to 0$ when $n \to \infty$.

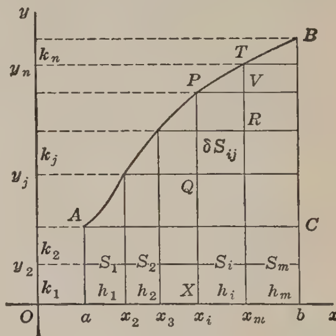

FIG. 101.

The parts into which S is thus divided are of the type δS considered in § 149. But we may ignore the triangles PVT when finding $\int_S f(x, y)dS$; for their contribution to $\Sigma f(x, y)\,\delta S$ is numerically $< M \cdot h \cdot CB$, where M is the greatest value of $|f(x, y)|$ in S and h is the greatest h_i; it therefore $\to 0$ when $m \to \infty$. Hence if m_{ij} and M_{ij} denote the least and

greatest values of $f(x, y)$ in the rectangle (QR) whose sides are h_i and k_j, we have, by § 149,

$$\int_S f(x, y)dS = \lim \Sigma m_{ij}h_ik_j = \lim \Sigma M_{ij}h_ik_j \qquad (1)$$

1. The portions of the sums Σ in (1) which belong to the strip S_i have the factor h_i in common and may be written $h_i\Sigma_i m_{ij}k_j$ and $h_i\Sigma_i M_{ij}k_j$.

Since $m_{ij} \leqq f(x_i, y_j) \leqq M_{ij}$, we have

$$\Sigma_i m_{ij}k_j \leqq \Sigma_i f(x_i, y_j)k_j \leqq \Sigma_i M_{ij}k_j \qquad (2)$$

But in the sum $\Sigma_i f(x_i, y_j)k_j$, x_i is a constant; the k_j's are a set of parts, which $\to 0$, of the segment x_iP of the line $x = x_i$ between $y = 0$ and $y = \phi(x_i)$; and $f(x_i, y_j)$ is a value of the y-function $f(x_i, y)$ in the part k_j. Hence when $n \to \infty$, $\Sigma_i f(x_i, y_j)\, k_j \to \int_0^{\phi(x_i)} f(x_i, y)dy$, § 124. Therefore, by (2),

$$\Sigma_i m_{ij}k_j \leqq \int_0^{\phi(x_i)} f(x_i, y)dy \leqq \Sigma M_{ij}k_j \qquad (3)$$

2. Multiply each member of (3) by h_i. Then apply the resulting inequalities to each of the strips $S_1, S_2, \cdots S_m$ and add corresponding members of the results. We get

$$\Sigma m_{ij}h_ik_j \leqq \Sigma h_i \int_0^{\phi(x_i)} f(x_i, y)dy \leqq \Sigma M_{ij}h_ik_j \qquad (4)$$

By (1) the limit of the first and last members of (4) is $\int_S f(x, y)dS$, and by § 151 the limit of the second member is $\int_a^b \int_0^{\phi(x)} f(x, y)dy\, dx$. Hence

$$\int_S f(x, y)dS = \int_a^b \int_0^{\phi(x)} f(x, y)dy\, dx \qquad (5)$$

This formula also holds good when the curve $y = \phi(x)$ has any finite number of turning points between $x = a$ and $x = b$; for parallels to Oy through these points will divide the corresponding space S into pieces of the type just considered.

When $f(x, y)$ is 1, the formula (5) becomes

$$\text{Area } S = \int_a^b \int_0^{\phi(x)} dy\, dx \qquad (6)$$

153. Computation of volumes. Let K denote a solid bounded below by the space S of § 152, above by a continu-

ous surface $z = f(x, y)$ which does not cut S, and laterally by the planes $x = a$, $x = b$, $y = 0$, and the cylindrical surface formed by parallels to Oz through the points of the curve $y = \phi(x)$.

The portion of K which rests on each part δS of S is included between two cylinders whose volumes are the least and greatest values of $z\delta S$ in δS. Hence, § 138, the volume V of K is $\int_S z\,dS$, and therefore, § 152,

$$V = \int_a^b \int_0^{\phi(x)} z\,dy\,dx \qquad (1)$$

The first integration gives the area $A(x)$ of the section of K by the plane $x = x$, the second evaluates $\int_a^b A(x)dx$. (Compare § 141.)

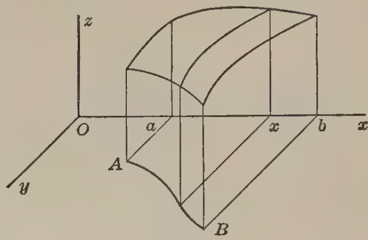

Fig. 102.

Example. For the solid bounded by $x = 2$, $x = 4$; $y = 0$, $y = x$; $z = 0$, $z = x^2 + 3\,y^2$, we have

$$V = \int_2^4 \int_0^x (x^2 + 3\,y^2)dy\,dx = \int_2^4 (x^2y + y^3)_{y=0}^{y=x}\,dx = 2\int_2^4 x^3\,dx = 120$$

For a solid standing on a space S bounded by $x = a$, $x = b$ and two curve arcs $y = \phi(x)$, $y = \psi(x)$ which do not meet between $x = a$ and $x = b$, we have

$$V = \int_a^b \int_{\phi(x)}^{\psi(x)} z\,dy\,dx \qquad (2)$$

EXERCISE XXXII

The student is advised to read §§ 204, 205 and Exs. 2 and 3 in § 213 before doing the following examples.

1. Compute the following and indicate the areas which they represent.

1. $\int_1^2 \int_{-x^2}^{x} dy\,dx$ 2. $\int_2^3 \int_y^{1-y} dx\,dy$ 3. $\int_0^1 \int_{-2x}^{x+x^2} dy\,dx$

2. Compute the following and indicate the volumes which they represent.

1. $\int_{-1}^2 \int_2^3 (2\,x + 3\,y + 6)dy\,dx$ 2. $\int_2^3 \int_{\sqrt{2}}^{x} xy\,dy\,dx$

3. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} (x + y)dy\,dx$

3. Show that under the conditions of § 150,

$$\int_a^b\int_c^d f(x, y)dy\,dx = \int_c^d\int_a^b f(x, y)dx\,dy$$

4. Find the volume of one of the wedges cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = x + y$.

5. Find the volume of the solid bounded by the planes $y = x$, $y = 2x$, $x = 1$, $z = 0$ and the surface $z = xy$.

6. Find the volume of the solid bounded by the planes $z = x + y$, $z = 0$, $x = 2$ and the cylindrical surface $y^2 - 4x = 0$.

7. Find the volume of the solid bounded by the surfaces $x^2 + 3y^2 = z$, $x^2 + y^2 = 2x$ and the plane $z = 0$.

8. Find the volume of the solid bounded by the surfaces $y = x^2$, $x = y^2$, and the planes $z = x$, $z = 2x$.

9. Find volume bounded by the cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$, in the first octant.

10. Find the volume bounded by the planes $x + 3y + 2z = 6$, $x = 0$, $y = 0$, $z = 0$.

11. Find the volume bounded by the surface $x^{1/2} + y^{1/2} + z^{1/2} = 1$, and the planes $x = 0$, $y = 0$, $z = 0$.

12. Set up the integral for the volume of the solid bounded by $z = x^2 + y^2$ and $z = 2x$ [Eliminating z gives $x^2 + y^2 = 2x$, which bounds the space in the xy-plane above which the solid lies.]

13. Let S be the space OAB bounded by the curve arc $r = \phi(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$. By rays from O and circles about O as center, divide S into parts δS. Show that $\delta S = PQTR = r\delta\theta\delta r$, where $r = (OP + OR)/2$ and therefore belongs to points in δS. Then show, as in § 152, that if $f(\theta, r)$ be continuous in S, we have

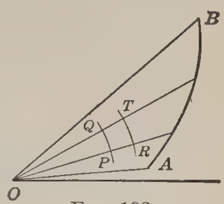

FIG. 103.

$$\int_S f(\theta, r)dS = \int_\alpha^\beta\int_0^{\phi(\theta)} f(\theta, r)r\,dr\,d\theta$$

14. Show that the volume of the solid bounded by $z = 0$, $z = x + 2y + 8$ and the right cylinder through the cardioid $r = 1 - \cos\theta$ is given by $2\int_0^\pi\int_0^{1-\cos\theta}[r(\cos\theta + 2\sin\theta) + 8]r\,dr\,d\theta$.

15. Show that the volume of the solid bounded by the sphere $x^2 + y^2 + z^2 = a^2$ and the right cylinder through the circle $r = a\cos\theta$

is $4\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}(a^2 - r^2)^{1/2}r\,dr\,d\theta = \frac{4a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right)$.

TRIPLE INTEGRALS

154. The integral $\int_S(x, y, z)dS$. Let $F(x, y, z)$ denote a one-valued function of the variables x, y, z in a given closed region S in space, its boundary included.

The function $F(x, y, z)$ is said to be *continuous at the point* (a, b, c) in S, if $F(a, b, c)$ has a finite value and if $F(x, y, z) \to F(a, b, c)$ when $(x, y, z) \to (a, b, c)$ in S. It is said to be *continuous in* S if it is continuous at all points of S.

By a repetition of the argument in §§ 148, 149, we may prove the theorem:

Let a region S in which $F(x, y, z)$ is continuous be divided into parts each of which can be inclosed in a sphere of radius ρ. Let δS denote any one of these parts and also its volume, and let (x, y, z) denote any point in δS. When $\rho \to 0$, the sum $\Sigma F(x, y, z)\delta S$ approaches a definite limit, and the value of this limit is independent of the mode of partition of S.

We represent $\lim \Sigma F(x, y, z)\delta S$ by the symbol $\int_S F(x, y, z)dS$, read "integral $F(x, y, z)$ over S." Hence by definition,

$$\int_S F(x, y, z)dS = \lim \Sigma F(x, y, z)\delta S \tag{1}$$

Let S be the solid K in Fig. 102. If the upper boundary surface $z = f(x, y)$ is met by parallels to the xy-plane in single points only or may be divided into portions of which this is true, it can be proved by an argument like that in § 152, but based on the division of S into parts by sets of planes parallel to the yz-, zx-, and xy-planes, that for this S

$$\begin{aligned}\int_S F(x, y, z)dS &= \int_a^b\int_0^{\phi(x)}\int_0^{f(x, y)} F(x, y, z)dz\,dy\,dx \\ &= \int_a^b\left[\int_0^{\phi(x)}\left(\int_0^{f(x, y)} F(x, y, z)dz\right)dy\right]dx \qquad (2)\end{aligned}$$

In the first integration (with respect to z), x and y play the rôles of constants, and in the second (with respect to y), x plays this rôle. When $F(x, y, z) = 1$, (2) becomes a formula for the volume of the solid K.

EXAMPLE 1. Find $\int_S x\,dS$ when S is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 4$.

We have $\int_S x\,dS = \int_0^4\int_0^{4-x}\int_0^{4-x-y} x\,dz\,dy\,dx$

$$=\int_0^4\int_0^{4-x}[xz]_{z=0}^{z=4-x-y}dy\,dx$$

$$=\int_0^4\int_0^{4-x}(4x - x^2 - xy)dy\,dx$$

$$=\int_0^4\left[4xy - x^2y - \frac{xy^2}{2}\right]_{y=0}^{y=4-x}dx$$

$$=\frac{1}{2}\int_0^4(16x - 8x^2 + x^3)dx = 10\tfrac{2}{3}.$$

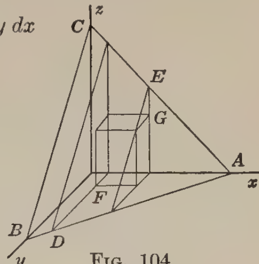

FIG. 104

Divide OA, OB, OC into parts δx, δy, δz, and through the points of division take planes parallel to the planes $x = 0$, $y = 0$, $z = 0$. We thus divide $OABC$ into parts δS of volume $\delta x\,\delta y\,\delta z$ (except those along ABC whose sum $\to 0$). The successive integrations correspond to forming $\Sigma x\delta S$ for the δS's in (1) a column (FG) parallel to Oz, (2) a slice (DE) parallel to Oyz, (3) the whole of $OABC$.

EXAMPLE 2. Find (1) $\int_0^a\int_0^b\int_0^c(x^2 + y^2 + z^2)dz\,dy\,dx$

(2) $\int_2^3\int_1^x\int_0^x x\,dz\,dy\,dx$

(3) $\int_0^a\int_0^x\int_0^{x+y} xyz\,dz\,dy\,dx$

EXAMPLE 3. Express $\int_S F(x, y, z)\,dS$ as a triple integral when S is the sphere whose equation is $x^2 + y^2 + z^2 - 2x = 0$.

XVII. MEAN VALUES

155. Mean values. 1. Let $y = f(x)$ be continuous in the x-interval (a, b). Divide (a, b) into n *equal* parts, of length h, and let $y_1, y_2, \cdots$ be any values taken by y in the first, second, $\cdots$ of these parts. The arithmetic mean of the numbers $y_1, y_2, \cdots$, namely

$$(y_1 + y_2 + \cdots + y_n)/n \tag{1}$$

will approach a definite limit when $n \to \infty$. For (1) equals

$$(y_1h + y_2h + \cdots + y_nh)/nh \tag{2}$$

But $nh = b - a$, and $\lim\limits_{n\to\infty} (y_1h + y_2h + \cdots + y_nh) = \int_a^b y\,dx$. Hence

$$\lim_{n\to\infty} \frac{\Sigma y_i}{n} = \frac{\int_a^b y\,dx}{b - a} \tag{3}$$

We call (3) the *mean value of* $y = f(x)$ *with respect to* x *over the range* (a, b).

EXAMPLE 1. The mean value, with respect to x, of the ordinates of the parabola $y = x^2$ between $x = 0$ and $x = 2$ is $[\int_0^2 x^2\,dx]/2 = 4/3$.

2. The notion of mean value is readily extended to functions of two or more variables. Thus, let S denote any closed region in the xy-plane and also its area; the mean value of $z = f(x, y)$ with respect to x, y over the range S is

$$\bar{z} = [\int_S f(x, y)\,dS]/S \tag{4}$$

We think of S as divided into equal parts of the type δS, § 149, choose a value taken by $z = f(x, y)$ in each of these parts and then proceed as in 1. In general,

The mean value of a function of one or more variables with respect to these variables over a given range is the integral of the function over the range, divided by the measure of the range.

EXAMPLE 2. Let R be the rectangle bounded by $x = 0$, $x = a$, $y = 0$, $y = b$. The mean value of the squares of the distances of the points of R from O is $[\int_0^a \int_0^b (x^2 + y^2)\,dy\,dx]/ab = (a^2 + b^2)/3$.

3. Multiplying (4) by S, we obtain

$$\bar{z}S = \int_S z\,dS \tag{5}$$

Suppose S to be divided into m parts $\Delta S_1, \Delta S_2, \cdots \Delta S_m$ of *any* kind, as for example, into strips by parallels to Oy: then

$$\int_S z\,dS = \int_{\Delta S_1} z\,dS + \int_{\Delta S_2} z\,dS + \cdots + \int_{\Delta S_m} z\,dS \tag{6}$$

Let $\bar{z}'_1, \bar{z}'_2, \cdots$ denote the mean values of z in $\Delta S_1, \Delta S_2, \cdots$. Then $\int_{\Delta S_1} z\,dS = \bar{z}'_1\,\Delta S_1$, $\int_{\Delta S_2} z\,dS = \bar{z}'_2\,\Delta S_2, \cdots$, by (5). Hence

$$\bar{z}S = \bar{z}'_1\,\Delta S_1 + \bar{z}'_2\,\Delta S_2 + \cdots + \bar{z}'_m \Delta S_m \tag{7}$$

The product of mean value by measure of range for the whole is the sum of such products for the parts.

The relation (6) continues to hold good when $m \to \infty$ and every $\Delta S_i \to 0$. Hence

$$\bar{z}S = \int_S \bar{z}'\,dS \tag{8}$$

where the differential dS represents ΔS, and $\bar{z}'$ the mean value of z in ΔS.

EXAMPLE 3. Find the mean distance of the points of the surface S of a circle of radius a from its center O. Divide S into circular rings. Let ΔS be any one of these rings, r its inner radius and δr its breadth. The mean distance r' of the points of ΔS from O is between r and $r + \delta r$; and $\Delta S = 2\pi r''\delta r$, where $r'' \doteq r + \delta r/2$, § 139. Hence the required mean distance is $[\int_0^a r\,2\pi r\,dr]/a^2\pi = (2/3)a$, § 145.

EXERCISE XXXIII

1. Show that the mean value of $\sin^2\theta$ with respect to θ between $\theta = 0$ and $\theta = \pi/2$ is $1/2$.

2. The circle of radius a about O may be represented by $x^2 + y^2 = a^2$ or by $x = a\cos\theta$, $y = a\sin\theta$. Hence show that the mean value of the ordinates of the first quadrant is $a\pi/4$ when based on the division of the radius along Ox into equal parts, and that it is $2a/\pi$ when based on the division of the arc into equal parts.

3. The final velocity of a body falling from rest is v_1. Show that its mean velocity during its motion was $v_1/2$ with respect to time, but $(2/3)v_1$ with respect to distance.

4. Show that the mean distance from O of the points of the surface of the circle $r = a \cos \theta$ is $16\, a/9\pi$. See p. 173, Ex. 13.

5. Derive the mean value theorem as stated in § 129 (6) from § 155 (3).

6. If a denote the earth's radius and z the distance of a point P of the earth's surface from the plane of the equator, then the latitude of P is $\sin^{-1}(z/a)$. Show that the mean latitude of all points north of the equator is $\frac{1}{a}\int_0^a \sin^{-1}\frac{z}{a}\,dz = \frac{\pi}{2} - 1$ radians (32.7°).

7. Let S_1 and S_2 be two regions which do not overlap, and let M_1, M_2, and M be the mean values of z in S_1, S_2, and $S_1 + S_2$. Show that $M(S_1 + S_2) = M_1S_1 + M_2S_2$.

8. By Ex. 7, find the mean distance from O of points between the circles $r = a$ and $r = a \cos \theta$.

9. A number c is divided at random into two parts; show that the mean value of their product is $c^2/6$.

10. Find the mean value of the square of the distance of a point of a square from its center.

11. A number c is divided at random into three parts; show that the mean value of their product is $c^3/60$.

CENTROIDS AND MASS CENTERS

156. Centroids. 1. Let S denote a given curve arc, surface, or solid; also its numerical measure. The point C whose coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ are the mean values of the coordinates x, y, z of the points of S is called the *centroid*[1] of S. By § 155 (5),

$$\bar{x}S = \int_S x\,dS \qquad \bar{y}S = \int_S y\,dS \qquad \bar{z}S = \int_S z\,dS \quad (1)$$

Here dS stands for any one of a set of equal parts of S of the type δS, § 149, and (x, y, z) for any point in δS. When S is in the xy-plane, $\bar{z} = 0$.

[1] It can be shown that the position of the centroid as thus defined, relative to the points of S, is independent of the choice of the coordinate axes Ox, Oy, Oz.

EXAMPLE 1. Find the centroid of the first quadrant of the circle $x = a\cos\theta$, $y = a\sin\theta$.

$$\bar{x}\frac{a\pi}{2} = \int_{\theta=0}^{\theta=\pi/2} x\,ds = \int_0^{\pi/2} a\cos\theta \cdot a\,d\theta = a^2 \quad \therefore \bar{x} = \frac{2a}{\pi}. \quad \text{And } \bar{y} = \bar{x}.$$

EXAMPLE 2. Find the centroid of the surface of the first quadrant of the circle $x^2 + y^2 = a^2$.

$$\bar{x}\frac{a^2\pi}{4} = \int_0^a \int_0^{(a^2-x^2)^{1/2}} x\,dy\,dx = \int_0^a x(a^2 - x^2)^{1/2}\,dx = \frac{a^3}{3} \quad \therefore \bar{x} = \frac{4a}{3\pi}.$$

And $\bar{y} = \bar{x}$.

EXAMPLE 3. Find the centroid of the solid bounded by the planes $x = 2$, $y = x$, $y = 2x$, $z = 0$, $z = x$.

$\bar{z}\int_0^2\int_x^{2x}\int_0^x dz\,dy\,dx = \int_0^2\int_x^{2x}\int_0^x z\,dz\,dy\,dx$, which gives $\bar{z} = 3/4$.

A similar reckoning gives $\bar{x} = 3/2$ and $\bar{y} = 9/4$.

EXAMPLE 4. Find the centroid of the region bounded by the curves
$$2y = 2x - x^2,\ y = x^2 - 2x.$$

EXAMPLE 5. Find the centroid of the solid bounded by the cylindrical surfaces $y = x^2$, $y^2 = x$, and the planes $z = 0$, $z = y$.

2. If S is symmetric with respect to a certain plane E, its centroid is in E. For refer the points of S to coordinate axes in which the xy-plane coincides with E. Then in the sum $\Sigma z\delta S$ every two terms belonging to parts δS which are symmetric with respect to E will cancel each other; hence $\int_S z\,dS = 0$ and therefore $\bar{z} = 0$.

In like manner, if S is symmetric with respect to a line l, its centroid is in l; if symmetric with respect to a point C, its centroid is C.

Thus the centroid of a square or of an ellipse is its geometric center.

3. Let S denote the portion of a given solid between the planes $x = a$ and $x = b$, and $A(x)$ the area of the section A_x of S by a plane perpendicular to Ox at the distance x from O [Fig. 92, p. 158]. Then

$$\bar{x}S = \int_a^b xA(x)dx \tag{2}$$

For by (1), $\bar{x}S = \int_a^b x[\int\int_{A_x} dz\,dy]dx = \int_a^b xA(x)dx$

Thus for the solid got by revolving about Ox the space bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = a$, $x = b$, we have

$$\bar{x}\int_a^b y^2\,dx = \int_a^b xy^2\,dx \qquad \bar{y} = 0 \qquad z = 0 \qquad (3)$$

EXAMPLE 1. Find the centroid of the right circular cone of altitude a and radius b. We get the cone by revolving about Ox the part of the line $y = (b/a)x$, between $x = 0$ and $x = a$. Hence

$$x\int_0^a \frac{b^2}{a^2}x^2\,dx = \int_0^a \frac{b^2}{a^2}x^3\,dx, \text{ which gives } \bar{x} = \tfrac{3}{4}a. \quad \text{And } \bar{y} = 0, \bar{z} = 0.$$

EXAMPLE 2. Show that the centroid of a solid hemisphere of radius a is on the central radius at the distance $(3/8)\,a$ from the bounding plane.

EXAMPLE 3. Find the centroids of the solids, p. 157, Exs. 4–9, 17–22.

EXAMPLE 4. Find the centroid of the solid bounded by the surface $4\,x = y^2 + 2\,z^2$ and the plane $x = 4$.

157. Theorems of Pappus. 1. Suppose a curve arc AB of length s in the xy-plane to be revolved about Ox. By § 144 (1) and § 156 (1), we have

$$\text{Area gen'd by } AB = 2\,\pi\int y\,ds = 2\,\pi\bar{y}\int ds = 2\,\pi\bar{y}\cdot s \qquad (1)$$

The area generated by the arc equals the length of the arc times the perimeter of the circle described by its centroid.

2. Suppose a closed region S in the xy-plane, which does not cut Ox, to be revolved about Ox. Divide S into parts of the type $\delta S = \delta x\,\delta y$. By § 139 (2) the volume of the solid generated by δS is $2\,\pi y\delta S$, where y belongs to a point in δS. Hence, § 156 (1),

$$\text{Volume gen'd by } S = 2\,\pi\int_S y\,dS = 2\,\pi\bar{y}\cdot S \qquad (2)$$

The volume generated by S equals the area of S times the perimeter of the circle described by the centroid of S.

EXAMPLE. For the torus got by revolving the circle $x^2 + (y - b)^2 = a^2$, $a < b$, about Ox we have area $= 2\,\pi b\cdot 2\,\pi a = 4\,\pi^2 ab$, and volume $= 2\,\pi b\cdot a^2\pi = 2\,\pi^2 a^2 b$.

158. Mass center. Let S denote a physical solid and m its mass. The *mass center* of S is the point whose coordinates

$\bar{x}$, $\bar{y}$, $\bar{z}$ are the mean values of the coordinates x, y, z of the points of S corresponding to a division of S into parts δS of equal mass δm. We therefore have

$$m\bar{x} = \int_S x\,dm \qquad m\bar{y} = \int_S y\,dm \qquad m\bar{z} = \int_S z\,dm \qquad (1)$$

When the ratio $\delta m/\delta S$ has the same value c for all parts δS, S is said to be *homogeneous and of density* c. In this case $dm = c\,dS$, $m = cS$; hence the formulas (1) reduce to the formulas § 156 (1), that is, the mass center coincides with the centroid of S.

Suppose S to be non-homogeneous. Let P be any given point of S, δS a part of S containing P, and δm its mass. When $\delta S \to 0$, the ratio $\delta m/\delta S$ approaches a limit μ; we call μ the *density of* S *at* P. If μ be given for every point P of S by some continuous function of the coordinates x, y, z, we obtain formulas for $\bar{x}$, $\bar{y}$, $\bar{z}$ by setting $dm = \mu\,dS$, $m = \int_S \mu\,dS$ in (1). All that has been said holds good also when a mass m is supposed to be distributed over a given surface or line S.

EXAMPLE. Find the mass center of the square bounded by $x = 0$, $x = a$, $y = 0$, $y = a$ when the density at a point varies as the square of the distance from the origin.

Here $\mu = c(x^2 + y^2)$, where c is a constant. Hence

$\bar{x}\int_0^a \int_0^a c(x^2 + y^2)dy\,dx = \int_0^a \int_0^a xc(x^2 + y^2)dy\,dx$ which gives $\bar{x} = (5/8)a$. And $\bar{y} = \bar{x}$.

EXERCISE XXXIV

1. Show that if the origin is the mass center of S, then $\int_S x\,dm = 0$, $\int_S y\,dm = 0$, $\int_S z\,dm = 0$.

2. Show that the centroid of the curved surface of a hemisphere of radius a is at the distance $(1/2)\,a$ from the bounding plane. Find the centroid of the total surface.

3. Find the centroid of the total surface of a right circular cone.

4. Find the mass center of a semicircle when the density varies as (1) the distance from the bounding diameter (2) the distance from the center of the circle.

5. Find the centroid of the solid got by revolving the figure bounded by $y = x^{1/2}$, $y = -1$, $x = 0$ and $x = 1$ about $x = 1$.

6. Find the centroid of the space bounded by Ox and an arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. Also of the solid got by revolving this space about Ox.

7. For the S bounded by $r = f(\theta)$, $\theta = 0$, $\theta = \beta$, show that

$$\bar{x}S = \int_0^\beta \int_0^{f(\theta)} xr\,dr\,d\theta, \quad \bar{y}S = \int_0^\beta \int_0^{f(\theta)} yr\,dr\,d\theta,$$

where $x = r\cos\theta$, $y = r\sin\theta$.

8. Find the centroid of the S between Ox and the upper half of $r = a(1 - \cos\theta)$.

9. Show that the volume of the solid got by revolving the S of Ex. 7 about Ox is $2\pi \int_0^\beta \int_0^{f(\theta)} yr\,dr\,d\theta$, and that for the centroid of this solid we have $\bar{x}\int_0^\beta \int_0^{f(\theta)} yr\,dr\,d\theta = \int_0^\beta \int_0^{f(\theta)} x\,yr\,dr\,d\theta$.

10. Apply the formulas in Ex. 9 to the solid got by revolving the S in Ex. 8 about Ox.

11. Show that the centroid of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ is $\left(\dfrac{a}{4}, \dfrac{b}{4}, \dfrac{c}{4}\right)$.

12. Find the volume and area of the solid got by revolving the rectangle bounded by $y - x = 0$, $y - x = 2$, $y + x = 0$, $y + x = 6$ about Ox.

13. Show that the mean distance of points in a sphere of radius a from a point O on its surface is $(6/5)a$; also that the mean distance of the points of the surface of the sphere from O is $(4/3)a$.

14. Show that the centroid of a triangle is the point of intersection of its medians.

15. Express the coordinates of the centroid of the perimeter of a triangle in terms of the lengths of the sides and the coordinates of their midpoints.

XVIII. MOMENTS OF INERTIA

159. Moment of inertia. Let l be a given axis, and P a particle of mass m at the distance r from l; the product mr^2 is called the *moment of inertia* of P about l.

Suppose that instead of being concentrated at a point P, the mass m is distributed through a line, surface or solid S. Divide S into parts of the type δS, § 149, multiply the mass δm of each part δS by the square of the distance r of any point in it from l, and add the products thus obtained. The resulting sum $\Sigma r^2 \delta m$ will approach the limit $\int_S r^2 dm$ when the parts $\delta S \rightarrow 0$. This limit is called the *moment of inertia* of m about l. Denoting it by I_l,

$$I_l = \int_S r^2 dm \tag{1}$$

When the mass in S is homogeneous and of unit density, (1) reduces to $I_l = \int_S r^2 dS$, which is called the moment of inertia of S itself about l.

When S is in a plane E and l is the perpendicular to E through a point O of E, we may replace the symbol I_l by I_O and call it the moment of inertia of the m in S about O.

EXAMPLE 1. To find the moment of inertia of a rod of length a about an axis perpendicular to the rod and through one of its end points, take the end point as origin and the line of the rod as Ox; we then have

$$I_O = \int_0^a x^2\, dx = a^3/3 = (a^2/3)m$$

EXAMPLE 2. For the space S between the parabola $y^2 = 4x$ and the line $y = x$, find the moment of inertia about (1) Ox, (2) Oy, (3) O.

$$I_x = \int_S y^2\, dS = \int_0^4 \int_x^{2\sqrt{x}} y^2\, dy\, dx = 64/5$$

$$I_y = \int_S x^2\, dS = \int_0^4 \int_x^{2\sqrt{x}} x^2\, dy\, dx = 64/7$$

$$I_O = \int_S (x^2 + y^2) dS = \int_S x^2\, dS + \int_S y^2\, dS = I_y + I_x = 768/35$$

EXAMPLE 3. The lengths of the edges OA, OB, OC of a rectangular parallelopiped are a, b, c; prove that its moment of inertia about the edge OC is

$$I_{OC} = \int_0^a \int_0^b \int_0^c (x^2 + y^2) dz\, dy\, dx = abc \frac{a^2 + b^2}{3}$$

EXAMPLE 4. Show that the moment of inertia of the circumference of a circle of radius a about its center O is $a^2 m$ $(m = 2\pi a)$. Also, by taking rectangular axes Ox, Oy, and using the relation $I_O = I_x + I_y$ (Ex. 2), show that its moment of inertia about a diameter is $(a^2/2)m$.

160. Radius of gyration. Let k^2 denote the ratio I_l/m; then

$$I_l = k^2 m \tag{2}$$

Hence k denotes the distance from l at which a particle of mass m has the same moment of inertia that the mass m in S has; k is called the *radius of gyration* of the mass m in S about l.

Thus, in § 159, Ex. 1, we have $k^2 = a^2/3$, and in Ex. 3 we have $k^2 = (a^2 + b^2)/3$.

From (1) and (2), we have $k^2 m = \int_S r^2\, dm$. Hence, § 155, k^2 is the mean value of r^2 with respect to m for the points of S. But from this it follows that the formula $I_l = \int_S r^2\, dm$ is true when dm stands for the mass of a part of S of *any* type ΔS, and r for the radius of gyration of this dm, § 155, 3.

EXAMPLE 1. Thus, to find the moment of inertia of a circular plate S of radius a about its center O, we may divide S into circular rings ΔS. For the ΔS of inner radius ρ and breadth $d\rho$, we may set $dm = 2\pi\rho d\rho$ and $r = \rho$ in $\int_S r^2 dm$, which gives

$$I_O = \int_0^a 2\pi\rho^3 d\rho = \frac{\pi a^4}{2} = \frac{a^2}{2}\pi a^2 = \frac{a^2}{2} m$$

The moment of inertia of the plate about a diameter is $(a^2/4)m$.

EXAMPLE 2. For the surface $S = abBA$ in Fig. 105, show that

1. $I_x = \int_a^b \frac{y^2}{3} \cdot y\, dx$ **2.** $I_y = \int_a^b x^2 \cdot y\, dx$

For the solid V_x got by revolving S about Ox, that **3.** $I_x = \int_a^b \frac{y^2}{2} \pi y^2\, dx$

For the solid V_y got by revolving S about Oy, that **4.** $I_y = \int_a^b x^2 \cdot 2\pi xy\, dx$

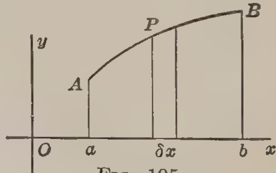

FIG. 105.

EXAMPLE 3. Apply the formulas 1.–4. of Ex. 2. to the S bounded by Ox, $y = x$, and $x = 2$.

EXAMPLE 4. Show that the moment of inertia of a sphere of radius a about a diameter is $(2/5)a^2m$, $m = (4/3)\pi a^3$.

EXAMPLE 5. For the surface of the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(m = ab\pi)$ show that

$$I_x = \frac{4}{3}\int_0^a y^3\,dx = \frac{b^2}{4}m \qquad I_y = \frac{4}{3}\int_0^b x^3\,dy = \frac{a^2}{4}m \qquad I_O = \frac{a^2+b^2}{4}m$$

EXAMPLE 6. Referring to § 141, Ex. 1, show that the moment of inertia of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ about the axis Oz is

$$I_z = 2\int_0^c \frac{a^2+b^2}{4}ab\pi\left(1 - \frac{z^2}{c^2}\right)^2 dz = \frac{a^2+b^2}{5}m, \left(m = \frac{4}{3}\pi abc\right)$$

161. Theorem. *Let m be a mass distributed through a space S, l any given axis, l' the axis parallel to l through the mass center of S, and a the distance between l and l'; then*

$$I_l = I_{l'} + a^2m \qquad (\therefore\ k_l^2 = k_{l'}^2 + a^2) \qquad (3)$$

For take the mass center of S as origin, l' as Oz, and the perpendicular to l through O as Ox. Let $P(x, y, z)$ be any point of S, and let r and r' denote the distances PD and PE of P from l and l'.

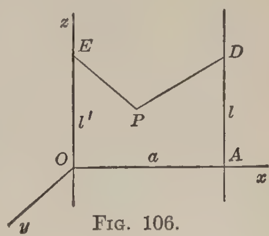

FIG. 106.

$$PD^2 = (x-a)^2 + y^2$$
$$PE^2 = x^2 + y^2$$
$$\therefore\ r^2 = (x^2+y^2) + a^2 - 2\,ax = r'^2 + a^2 - 2\,ax$$
$$\therefore\ \int_S r^2\,dm = \int_S r'^2\,dm + a^2m - 2\,a\int_S x\,dm$$

This is the formula (3); for, O being the mass center, $\int_S x\,dm = 0$, § 158 (1).

EXAMPLE 1. Find the moment of inertia of a circle of radius a about a tangent l. Let l' be the diameter parallel to l; then $I_l = I_{l'} + a^2m$ $= \dfrac{a^2}{4}m + a^2m = \dfrac{5}{4}a^2m$.

EXAMPLE 2. Find the moment of inertia of a cylinder of radius a and length $2\,b$ with respect to a line through the center and perpendicular to the axis of the cylinder.

Let O be the center and Ox the axis of the cylinder. The moment of inertia about Oz of the section of the cylinder by a plane perpendicular to Ox at the distance x from O is $\left(\frac{a^2}{4} + x^2\right)a^2\pi$. Therefore

$$I_z = 2\int_0^b\left(\frac{a^2}{4} + x^2\right)a^2\pi dx = 2\,a^2\pi b\left(\frac{b^2}{3} + \frac{a^2}{4}\right)$$

EXAMPLE 3. Show that the moment of inertia of a cone of radius a and altitude b about a line through its vertex and perpendicular to its axis is

$$I_z = \int_0^b\left[\frac{a^2}{4\,b^2}\,x^2 + x^2\right]\frac{a^2}{b^2}\,x^2\pi\,dx = \frac{3}{20}\,m(4\,b^2 + a^2)$$

162. Kinetic energy of rotation. Let P be a particle of mass m at the distance r from an axis l and rotating about l with the angular velocity ω. Its speed is $v = r\omega$, § 88; hence its kinetic energy, namely $E = mv^2/2$, is

$$E = \tfrac{1}{2}\,mr^2\omega^2 = \tfrac{1}{2}\,I_l\,\omega^2$$

Suppose that the mass m is distributed through a body S which is rotating about l with the angular velocity ω. As in § 159, divide S into parts δS of mass δm. The kinetic energy of m is the sum of the kinetic energies of the parts δm, and the kinetic energy of the δm in each δS is between the least and greatest values of $r^2\delta m \cdot \omega^2/2$ for the points P of that δS. Hence the kinetic energy of the mass m in S is

$$E = \tfrac{1}{2}\,I_l\,\omega^2, \text{ where } I_l = \int_S r^2dm \tag{4}$$

EXAMPLE 1. For a thin homogeneous rod of length a and mass m which is rotating about an end point with the angular velocity ω, we have $E = \frac{1}{2}\,\frac{a^2}{3}\,m\omega^2$.

When the mass of the body S is expressed in pounds, and r and ω in feet and radians per second, the formula (4) gives E in "foot-poundals." If the kinetic energy be required in "foot-pounds," this result must be divided by $g = 32$, the acceleration due to gravity.

EXAMPLE 2. For a homogeneous solid wheel of 4 ft. radius, weighing 1000 lbs., and making 30 revolutions per minute, we have

$$E = \frac{1}{2}\,\frac{4^2}{2}\,\frac{1000}{32}\left(\frac{30}{60}\,2\,\pi\right)^2 = 1233 \text{ ft. lbs.}$$

EXERCISE XXXV

1. Find I_x, I_y, and I_O for the space S bounded by the parabolas $y = 2x^2$, $y^2 = 4x$.

2. Find I_z for the S bounded by the paraboloid $z = x^2 + 2y^2$ and the planes $z = 0$, $x = 0$, $x = 2$, $y = 0$, $y = 1$.

3. Find I_x, I_y, I_O for the triangle $y = x$, $y = 2x$, $x + y - 6 = 0$.

4. For the solid got by revolving the space bounded $y^2 = 4x$ and $x = 3$ about Ox find I_x, I_y.

5. Find the moment of inertia of a cylinder of radius a and altitude b about its axis.

6. Show that the moment of inertia of a cone of radius a and altitude b about its axis is $(3/10)ma^2$.

7. Find the moment of inertia of the perimeter of a square of side a about (1) a side, (2) the center, (3) a diagonal.

8. Show that the moment of inertia of a rectangle, of sides a, b, about a corner is $m\dfrac{a^2 + b^2}{3}$; about its center, $m\dfrac{a^2 + b^2}{12}$.

9. From Ex. 8 deduce that the moment of inertia of a rectangular parallelopiped of edges a, b, c about the edge c is $m\dfrac{a^2 + b^2}{3}$.

10. Show that the moment of inertia of a parallelogram of sides a, b and angle θ about the side a is $(mb^2 \sin^2 \theta)/3$.

11. Show that the moment of inertia of a triangle of base b and altitude h about the base is $mh^2/6$; about the line through the mass center parallel to the base, $mh^2/18$.

12. A right pyramid has a square base of side a, and its altitude $AD = h$. Show that its moment of inertia about AD is $ma^2/10$.

13. For the surface got by revolving the arc of $y = f(x)$ between $x = a$ and $x = b$ about Ox show that $I_x = 2\pi\int_{x=a}^{x=b} y^3\,ds$.

14. Show that the moment of inertia of a spherical surface of radius a about a diameter is $\frac{2}{3}ma^2$.

15. Find the moment of inertia of a solid sphere of radius a about a diameter by dividing it into spherical shells.

16. For the torus got by revolving the circle $x^2 + (y - b)^2 = a^2$, $b > a$, about Ox, Fig. 98, p. 163, show that $I_x = m(b^2 + \frac{3}{4}a^2)$.

17. For the surface of the torus in Ex. 16 show that $I_x = m(b^2 + \frac{3}{2}a^2)$.

18. For the area bounded by the curve $\rho = f(\theta)$ and the rays $\theta = \alpha$, $\theta = \beta$, show that $I_O = \iint_S \rho^3 d\theta d\rho = \int_\alpha^\beta \frac{\rho^4}{4} d\theta$.

19. Find I_O for the area of the cardioid $\rho = a(1 - \cos\theta)$.

20. Find the moment of inertia of a rod of length a about an end point when the density varies as the distance from this end point.

21. Find the moment of inertia of a sphere of radius a about a diameter when, ρ denoting the distance of a point from the center, the density varies as $a - \rho$.

22. The rim of a wheel weighs 500 pounds. It has 8 spokes, each 3 feet long and weighing 100 pounds. The wheel is making 40 revolutions per minute. Find its kinetic energy in foot-pounds.

23. Show that the moment of inertia of a hollow sphere of outer radius a and inner radius b with respect to a diameter is

$$\tfrac{2}{5} m(a^5 - b^5)/(a^3 - b^3).$$

XIX. ATTRACTION OF GRAVITATION

163. Attraction of gravitation. By the law of gravitation, a particle P of mass m attracts a particle P' of mass m' at the distance r from P with a force of intensity

$$F = k\frac{mm'}{r^2} \tag{1}$$

where k denotes a constant which can be found by experiment.

A particle of mass m' at the origin O is attracted by a mass m distributed through a space S not containing O with a force F whose x-, y-, z-components are

$$F_x = km'\int_S \frac{x}{r^3}\,dm \quad F_y = km'\int_S \frac{y}{r^3}\,dm \quad F_z = km'\int_S \frac{z}{r^3}\,dm \tag{2}$$

For divide S into parts, as in § 154. Let δS denote any one of these parts, δm its mass, P any point in δS, and r the length of OP. A particle of mass δm at P would attract the mass m' at O with the force $km'\delta m/r^2$ in the direction OP; and the x-component of this force, got by multiplying it by $\cos\,(OP, x)$, or x/r, is $km'(x/r^3)\delta m$. By the axioms of mechanics, the x-component of the attraction due to the δm distributed through δS is between the least and greatest values of this product $km'(x/r^3)\delta m$ for the points P of δS; and F_x is the sum of these x-components for all the parts δS. Hence $F_x = \lim \Sigma km'(x/r^3)\delta m = km'\int_S(x/r^3)dm$. Similarly for F_y and F_z.

The intensity of F is $[F_x^2 + F_y^2 + F_z^2]^{1/2}$; its direction is from O to S on the line $\dfrac{x}{F_x} = \dfrac{y}{F_y} = \dfrac{z}{F_z}$.

Example 1. With what force is a mass m' at O attracted by a homogeneous rod of density μ which extends from $x = a$ to $x = b$ on Ox?

$$F = F_x = km' \int_a^b \frac{\mu\, dx}{x^2} = km'\mu\left(\frac{1}{a} - \frac{1}{b}\right) = km' \frac{\mu(b-a)}{ab} = k\frac{m'm}{ab}.$$

Example 2. Find the x- and y-components of the force with which a particle of mass m' at O is attracted by a homogeneous rod of length l and density μ which stands on Ox at $x = c$. Also the direction of this force when $l = 3$ and $c = 4$.

$$F_x = km'\mu \int_0^l \frac{c\, dy}{(y^2 + c^2)^{3/2}} = k\frac{m'\mu}{c}\left[\frac{y}{(y^2+c^2)^{1/2}}\right]_0^l = k\frac{m'\mu l}{c(l^2+c^2)^{1/2}}$$

$$F_y = km'\mu \int_0^l \frac{y\, dy}{(y^2+c^2)^{3/2}} = km'\mu\left[\frac{1}{c} - \frac{1}{(l^2+c^2)^{1/2}}\right]$$

When $l = 3$ and $c = 4$, we have $(l^2 + c^2)^{1/2} = 5$ $\therefore F_x = km'\mu\,(3/20)$, $F_y = km'\mu(1/20)$ $\therefore F_y/F_x = 1/3$ $\therefore$ the direction of F makes the angle $\tan^{-1}(1/3)$ with Ox, and $F = km'\mu\sqrt{10}/20$.

Example 3. Show that the attraction of a homogeneous spherical surface S on a particle at a point outside of S is the same that it would be were the mass of S concentrated at its center. Show also that this theorem holds good for a homogeneous solid sphere.

1. Let the particle, of mass m', be at O, let μ be the density of S, and let $OC = c$ and $AC = a$. Since S is symmetric to Ox, we have $F_y = 0$, $F_z = 0$.

By planes perpendicular to Ox divide S into zones ΔS of altitude δx; $\Delta S = 2\pi a\, \delta x$ (p. 163, Ex. 2).

We have $OP = OQ$, and the equation of the circle $AQDB$ being $(x - c)^2 + z^2 = a^2$, we have $OQ^2 = x^2 + z^2 = 2cx - (c^2 - a^2)$. Hence show that

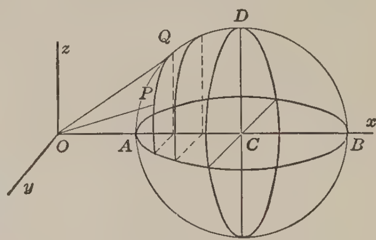

Fig. 107.

$$F = F_x = km'\,2\pi a\mu \int_{c-a}^{c+a} \frac{x\, dx}{[2cx - (c^2 - a^2)]^{3/2}} = km'\mu\frac{4a^2\pi}{c^2} = k\frac{m'm}{c^2}$$

2. Dividing the solid sphere of density μ bounded by S into shells of thickness $\delta\rho$, we find

$$F = F_x = km'\mu \int_0^a \frac{4\pi\rho^2}{c^2}\, d\rho = \frac{4}{3}\frac{\pi a^3}{c^2} km'\mu = k\frac{m'm}{c^2}$$

Example 4. Show that the attraction of a homogeneous spherical surface on a point inside the surface is 0.

Example 5. Show that each of two homogeneous solid spheres attracts the other with the force $k\,mm'/r^2$ where m, m' are their masses and r the distance between their centers.

Example 6. Find the attraction of a homogeneous hemispherical shell on a particle situated at the center of the sphere.

Example 7. Find the attraction of a homogeneous circular plate on a particle situated on the line through its center and pe pendicular to its plane.

Example 8. Find the attraction of a homogeneous cylinder on a particle situated on its axis produced.

Example 9. Let A be one of the points where a circle with center O and radius c cuts Ox, let B and C be any two points on the tangent to the circle at A, and let B' and C' be the points where the circle is cut by the lines OB and OC. Regarding the line segment BC and the arc $B'C'$ as homogeneous masses of unit density, show that their attractions on a particle of mass m' at O are the same in both magnitude and direction and therefore that the direction is along the bisector of the angle BOC. Derive the results in Ex. 2 from this theorem.

Example 10. Given any homogeneous rod AB and any point C not on the line of AB, find the direction of the force F with which AB attracts a particle at C.

XX. INFINITE SERIES

164. Infinite series. An expression of the form

$$u_1 + u_2 + \cdots + u_n + \cdots \tag{1}$$

where $u_1, u_2, \cdots, u_n, \cdots$ is any given never-ending sequence of numbers, is called an *infinite series.* Thus

$$1/2 + 1/2^2 + 1/2^3 + \cdots.$$

When we say that $u_1, u_2, \ldots$ is *given,* we mean that for every assigned value of n the value of u_n can be found. In the series that we shall meet, a formula for u_n in terms of n is given or indicated by the first few terms. Thus the series for which $u_n = n/(n^2 + 1)$ is $1/2 + 2/5 + \cdots$; and for the series $1/(2 \cdot 3) + 2/(3 \cdot 4) + \cdots$ we have $u_n = n/(n + 1)(n + 2)$.

The series (1) is often represented by the symbol Σu_n.

165. Convergence and divergence. Let S_n denote the sum of the first n terms of (1), so that $S_1 = u_1$, $S_2 = u_1 + u_2$, $\cdots$, $S_n = u_1 + u_2 + \cdots + u_n$. If, when $n \to \infty$, S_n approaches a finite limit, (1) is said to be *convergent,* and $\lim S_n$ is called its *sum;* but if S_n does not approach a finite limit, (1) is said to be *divergent.*

Thus, for the series $1/2 + 1/4 + 1/8 + \cdots$ the successive values of S_n are $1/2, 3/4, 7/8, \cdots$, and $S_n \to 1$; hence this series is convergent and its sum is 1.

On the other hand, the series $1 + 1 + 1 + \cdots$ and $1 - 1 + 1 - \cdots$ are divergent.

If the series (1) is convergent and S denotes its sum, we may write

$$S = S_n + R_n \tag{2}$$

where R_n denotes the sum of the part of (1) which follows the term u_n. Since $\lim S_n = S$, we have $\lim R_n = 0$.

Evidently S_n cannot approach a finite limit unless $u_n \to 0$. But $\lim u_n = 0$ does not ensure convergence. Thus in

$1 + 1/2 + 1/3 + \cdots$, we have $u_n = 1/n \to 0$; but this series is divergent, § 171.

If $u_1 + u_2 + \cdots (1)$ is convergent and has the sum S, then $cu_1 + cu_2 + \cdots$ (2) is convergent and has the sum cS; for $cu_1 + \cdots + cu_n = cS_n \to cS$. If (1) is divergent, so is (2).

166. Positive series. We call $u_1 + u_2 + \cdots$ a *positive series* when all its terms are positive. For such series we have the following theorems:

167. Theorem 1. *A positive series $u_1 + u_2 + \cdots$ is convergent if, as n increases, S_n remains always less than some finite number c.*

For since the series is positive, S_n continually increases as n increases. But it remains less than c. Hence it approaches a limit ($\leqq c$), § 5.

168. Theorem 2. *A positive series $u_1 + u_2 + \cdots$ is convergent if its terms are less than the corresponding terms of a positive series $a_1 + a_2 + \cdots$ known to be convergent.*

For if the sum of $a_1 + a_2 + \cdots$ be A, we have $S_n = (u_1 + u_2 + \cdots + u_n) < A$. Hence $u_1 + u_2 + \cdots$ is convergent, by Theorem 1. Similarly

169. Theorem 3. *A positive series $u_1 + u_2 + \cdots$ is divergent if its terms are greater than the corresponding terms of a positive series $b_1 + b_2 + \cdots$ known to be divergent.*

In the next two sections we consider the *test series* most frequently used in applying Theorems 2. and 3.

170. The geometric series. *The infinite geometric series $a + ar + ar^2 + \cdots$ is convergent when $|r| < 1$.*

For $S_n = a\dfrac{1 - r^n}{1 - r}$, and since $|r| < 1$, $\lim r^n = 0$.

Hence
$$\lim S_n = \frac{a}{1 - r}$$

Evidently $a + ar + ar^2 + \cdots$ is divergent when $|r| \geqq 1$.

171. The series $\Sigma 1/n^p$. *The series $1 + 1/2^p + 1/3^p + \cdots$ is convergent when $p > 1$, divergent when $p \lesseqgtr 1$.*

1. $p > 1$. Combining the two terms beginning at $1/2^p$, the four beginning at $1/4^p$, and so on, we obtain

$$1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots \tag{1}$$

whose terms, after the first, are less than those of

$$1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \cdots$$

$$= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \cdots = 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \cdots \tag{2}$$

But since $p > 1$ and therefore $1/2^{p-1} < 1$, the geometric series (2) is convergent. Hence (1) is convergent by Theorem 2.

2. $p = 1$. Combining the two terms ending at $1/4$, the four ending at $1/8$, and so on, we obtain

$$1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + \cdots \tag{3}$$

whose terms after the second are greater than those of

$$1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}) + \cdots = 1 + \tfrac{1}{2} + \tfrac{1}{2} + \cdots \tag{4}$$

But (4) is divergent. Hence (3) is divergent by Theorem 3.

3. $p < 1$. In this case $1 + 1/2^p + 1/3^p + \cdots$ is divergent since its terms are greater than those of the divergent series $1 + 1/2 + 1/3 + \cdots$.

Thus the series $1 + 1/2^2 + 1/3^2 + \cdots$ is convergent, but the series $1 + 1/\sqrt{2} + 1/\sqrt{3} + \cdots$ is divergent.

172. Applications of these test series. Consider the following examples:

Example 1. The series $x^2/(1 + x^2) + x^4/(1 + x^4) + \cdots$ converges when $|x| < 1$ since its terms are less than those of the geometric series $x^2 + x^4 + x^6 + \ldots$, which converges when $|x| < 1$.

Example 2. The series $1/(2 \cdot 3) + 1/(3 \cdot 4) + \cdots$ is convergent since its terms are less than those of the convergent series $1/2^2 + 1/3^2 + \cdots$.

Example 3. Is the series for which $u_n = (3n + 2)/(n^3 + 1)$ convergent or divergent?

Here $u_n = \dfrac{3n + 2}{n^3 + 1} = \dfrac{1}{n^2}\dfrac{3 + 2/n}{1 + 1/n^3}$; and $\dfrac{3 + 2/n}{1 + 1/n^3} \to 3$ when $n \to \infty$.

Hence after a certain term all terms of Σu_n are less than the corresponding terms of the series $\Sigma k/n^2$ where k is any assigned number > 3. Hence Σu_n is convergent.

By the reasoning in Ex. 3 it can be shown that if $u_n = f(n)/\phi(n)$, where $f(n)$ and $\phi(n)$ are integral expressions in n, the series Σu_n is convergent when the degree of $\phi(n)$ exceeds that of $f(n)$ by more than 1, divergent when this is not the case.

Example 4. In the following find u_n and test for convergence or divergence:

1. $\frac{1}{1} + \frac{1}{\sqrt{2^3}} + \frac{1}{\sqrt{3^3}} + \cdots$ 2. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$

3. $\frac{2}{1 \cdot 3} + \frac{4}{3 \cdot 5} + \frac{6}{5 \cdot 7} + \cdots$ 4. $\frac{2}{1 \cdot 2 \cdot 3} + \frac{4}{2 \cdot 3 \cdot 4} + \frac{6}{3 \cdot 4 \cdot 5} + \cdots$

5. $\frac{1}{a} + \frac{1}{2\,a + b} + \frac{1}{3\,a + b} + \cdots$

6. $\frac{2}{1 + 2\sqrt{2}} + \frac{3}{1 + 3\sqrt{3}} + \frac{4}{1 + 4\sqrt{4}} + \cdots$

Example 5. Is the series for which $u_n = (n^2 + n)^{1/2}/(n^4 + 1)^{1/3}$ convergent or divergent?

Example 6. The curve in Fig. 108 is the graph (for $x > 0$) of a positive decreasing function $y = f(x)$ which $\to 0$ when $x \to \infty$. The series $f(2) + f(3) + f(4) + \ldots$ is represented by the sum of the rectangles $2\,A_1$, $3\,A_2$, $\cdots$ and therefore converges if $\int_1^x f(x)dx \to$ a finite limit l when $x \to \infty$. The series $f(1) + f(2) + f(3) + \cdots$ is represented by the sum of rectangles $1\,B_2$, $2\,B_3$, $\cdots$ and therefore diverges if $\int_1^x f(x)dx \to \infty$ when $x \to \infty$. Fill in the details of this proof and show that the theorem of § 171 follows from it.

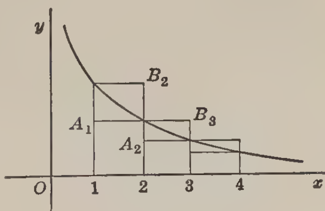

Fig. 108.

173. Theorem 4. *A positive series* $u_1 + u_2 + \cdots$ *is convergent if the ratio of each of its terms to the immediately preceding term is less than some number r which itself is less than* 1.

By hypothesis, $\frac{u_2}{u_1} < r, \; \frac{u_3}{u_2} < r, \; \frac{u_4}{u_3} < r, \cdots \frac{u_{n+1}}{u_n} < r, \cdots$

Multiplying the first of these inequalities by the second, the result by the third, and so on, and then clearing of fractions, we get

$$u_2 < u_1 r,\ u_3 < u_1 r^2,\ u_4 < u_1 r^3, \cdots u_{n+1} < u_1 r^n, \cdots$$

Hence the terms of $u_1 + u_2 + \cdots$ after the first are less than the corresponding terms of the convergent series $u_1 + u_1 r + u_1 r^2 + \cdots$. It is therefore convergent.

If $u_{n+1}/u_n \geqq 1$, the series $u_1 + u_2 + \cdots$ is divergent.

174. Corollary. *If, as $n \to \infty$, the ratio u_{n+1}/u_n approaches a limit l, the series $u_1 + u_2 + \cdots$ is convergent when $l < 1$, divergent when $l > 1$.*

For if $l < 1$, and r denote any number between l and 1, then after n reaches a certain value n' the ratio u_{n+1}/u_n will remain between l and r, that is, less than r; hence the series is convergent, § 173.

If $l > 1$, then after a certain value n' of n the ratio u_{n+1}/u_n will remain > 1; hence the series is divergent.

Example 1. Prove that $\frac{3}{4} + \frac{3 \cdot 5}{4 \cdot 7} + \cdots + \frac{3 \cdot 5 \cdots (2n+1)}{4 \cdot 7 \cdots (3n+1)} + \cdots$ is convergent.

Here $\lim \frac{u_{n+1}}{u_n} = \lim \frac{2n+1}{3n+1} = \lim \frac{2 + 1/n}{3 + 1/n} = \frac{2}{3}$, which is < 1.

A series in which $\lim u_{n+1}/u_n = 1$ may be convergent or divergent. It can be proved of such series that if u_{n+1}/u_n is reducible to the form

$$u_{n+1}/u_n = (n^p + an^{p-1} + \cdots)/(n^p + a'n^{p-1} + \cdots) \qquad (1)$$

the series is convergent when $a' - a > 1$, divergent when $a' - a \leqq 1$.

Thus in $\frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \cdots$, $\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+2} = \frac{n + 1/2}{n+1}$

Here $a' - a = 1 - 1/2 = 1/2$; hence the series is divergent.

EXAMPLE 2. Test the following series for convergence or divergence.

1. $\frac{1\cdot 2}{1\cdot 3}+\frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}+\frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7}+\cdots$ 2. $\frac{3}{4}+2(\frac{3}{4})^2+3(\frac{3}{4})^3+\cdots$

3. $\frac{4}{3}+\frac{4\cdot 8}{3\cdot 8}+\frac{4\cdot 8\cdot 12}{3\cdot 8\cdot 13}+\cdots$ 4. $\frac{3}{2}+\frac{3\cdot 7}{2\cdot 9}+\frac{3\cdot 7\cdot 11}{2\cdot 9\cdot 16}+\cdots$

5. $\frac{2}{3}+\frac{2\cdot 3}{3\cdot 4}+\frac{2\cdot 3\cdot 4}{3\cdot 4\cdot 5}+\cdots$ 6. $\frac{2}{5}+\frac{2\cdot 4}{5\cdot 8}+\frac{2\cdot 4\cdot 6}{5\cdot 8\cdot 11}+\cdots$

EXAMPLE 3. Show that $\frac{a}{1}+\frac{a(a+1)}{1\cdot 2}+\frac{a(a+1)(a+2)}{1\cdot 2\cdot 3}+\cdots$ is divergent if $a>0$.

EXAMPLE 4. Show by § 174 (1) that $\Sigma\, 1/n$ is divergent and $\Sigma\, 1/n^2$ convergent.

EXAMPLE 5. Show that $u_1+u_2+\cdots$ is convergent if $\sqrt[n]{u_n}<r<1$.

175. Series with positive and negative terms. The most important theorem respecting the convergence of such series is the following:

176. Theorem 5. *A series which has both positive and negative terms is convergent if the corresponding positive series is convergent.*

This is evident when the number of positive or of negative terms is finite; we therefore suppose that there are infinitely many terms of both kinds.

Let $u_1+u_2+\cdots$ (1) be the given series, and let $u'_1+u'_2+\cdots$ (2) denote (1) with the signs of all its negative terms changed. Also let S_n and S'_n be the sums of the first n terms in (1) and (2).

If there are p positive and q negative terms in S_n, and if P_p and $-N_q$ denote the sums of these terms, we have

$$S_n = P_p - N_q \qquad S'_n = P_p + N_q$$

By hypothesis, S'_n approaches the finite limit S' when $n\to\infty$. Hence P_p and N_q, both of which increase with n but remain less than S', also approach limits P and N, § 5, and therefore $S_n = P_p - N_q$ approaches the limit $P-N$. Hence $u_1+u_2+\cdots$ is convergent.

177. Absolute and conditional convergence. The converse of Theorem 5 is not true. For $S_n = P_p - N_q$ may also approach a finite limit when both P_p and N_q become infinite; then Σu_n is convergent, but $\Sigma u'_n$ divergent.

A convergent series is said to be *absolutely convergent* or *conditionally convergent* according as it remains convergent or becomes divergent when the signs of its negative terms, if any, are changed.

Thus in the next section it is shown that the series $1 - 1/2 + 1/3 - 1/4 + \cdots$ is convergent. But it is only conditionally convergent since the series $1 + 1/2 + 1/3 + \cdots$ is divergent, § 171.

178. Theorem 6. *An alternating series, that is, one whose terms are alternately postive and negative, is convergent if each term is numerically less than the one which precedes it and if the limit of the nth term is* 0.

Let $a_1 - a_2 + a_3 - \cdots$ be the series, $a_1, a_2, a_3, \cdots$, being positive. When n is even, we can group the terms in S_n in the two ways:

$$\begin{aligned} S_n &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{n-1} - a_n) & (1) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{n-2} - a_{n-1}) - a_n & (2) \end{aligned}$$

Since $a_1 > a_2 > a_3 > \cdots$, the bracketed expressions in (1) and (2) are all positive. Hence when n increases through even values, S_n increases, by (1), but remains less than a_1, by (2); it therefore approaches a limit l (§ 5). But S_{n+1} also approaches this limit l since $S_{n+1} = S_n + a_{n+1}$ and $a_{n+1} \to 0$. Hence $S_n \to l$ when $n \to \infty$ through the sequence 1, 2, 3, $\cdots$, and the series is convergent.

The expressions (1) and (2) also show that the sums $S_1 = a_1$, $S_2 = a_1 - a_2$, $S_3 = a_1 - a_2 + a_3$, and so on, are alternately greater and less than the sum S of the series, and therefore that S_n differs from S by less than a_{n+1}.

Thus the sum of the geometric series $1 - (.1)^2 + (.1)^4 - (.1)^6 + \cdots$ is 100/101, § 170; and S_3 exceeds 100/101 by less than $(.1)^6$ or .000001.

EXERCISE XXXVI

1. Which of the following series are divergent, conditionally convergent, absolutely convergent?

1. $\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} - \cdots$

3. $\frac{2}{3} - (\frac{2}{3})^2 + (\frac{2}{3})^3 - \ldots$

2. $\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt[3]{2}} + \dfrac{1}{\sqrt[4]{2}} - \cdots$

4. $\dfrac{1}{1-c} + \dfrac{1}{1-2c} + \dfrac{1}{1-3c} + \ldots$

2. Show that the sum S of $1 - 1/3! + 1/5! - 1/7! + \cdots$ lies between 1 and 5/6. To what decimal figure will S_3 represent S?

3. Add enough terms of $\dfrac{1}{5} + \dfrac{1}{3 \cdot 5^3} + \dfrac{1}{5 \cdot 5^5} + \cdots$ to find S to the third decimal figure.

4. Show that the remainder after n terms in $1 + x + x^2 + \cdots$ is $R_n = x^n/(1 - x)$. How great must n be to make $R_n < .1$ when $x = .9$? when $x = -.9$?

POWER SERIES

179. Power series in *x*. This name is given to a series of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \tag{1}$$

where x is a variable and $a_0, a_1, \cdots$ are constants.

Theorem. *If in the series $a_0 + a_1x + \cdots$ the ratio $|a_n/a_{n+1}|$ approaches a limit l, the series is absolutely convergent when $|x| < l$, divergent when $|x| > l$.*

For, by § 174, the positive series $|a_0| + |a_1x| + \cdots$ converges when

$$\lim \left|\frac{a_{n+1}x^{n+1}}{a_nx^n}\right| < 1, \quad \text{that is,} \quad \text{when } |x| < \lim \left|\frac{a_n}{a_{n+1}}\right|$$

Similarly, $a_0 + a_1x + \cdots$ diverges when $|x| > \lim |a_n/a_{n+1}|$.

We call l the *limit of convergence* of the series, and $(-l, l)$ its *interval of convergence*. The series may be convergent or divergent when $x = l$ or $-l$.[1]

[1] A series in which a_n/a_{n+1} does not approach a limit may also have a limit of convergence. Thus it is easily shown that the series $1 + 2x + x^2 + 2x^3 + \cdots$ converges when $|x| < 1$, diverges when $|x| \gtreqless 1$.

The following important series illustrate this theorem.

1. *The exponential series.* The series

$$1 + \frac{x}{1} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \tag{2}$$

is convergent for all finite values of x.

For $\lim \left|\frac{a_n}{a_{n+1}}\right| = \lim (n+1) = \infty$, that is, $l = \infty$.

2. *The logarithmic series.* The series

$$\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-)^{n-1}\frac{x^n}{n} + \cdots \tag{3}$$

is convergent when $|x| < 1$, divergent when $|x| > 1$.

For $\lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{n+1}{n} = \lim \left(1 + \frac{1}{n}\right) = 1$, that is, $l = 1$.

The series is convergent when $x = 1$, divergent when $x = -1$.

3. *The binomial series.* When m is not a positive integer,

$$1 + \frac{m}{1}x + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\cdots(m-n+1)}{n!}x^n + \cdots \tag{4}$$

is an infinite series. It converges when $|x| < 1$, diverges when $|x| > 1$. For

$$\frac{a_n}{a_{n+1}} = \frac{m(m-1)\cdots(m-n+1)}{n!} \div \frac{m(m-1)\cdots(m-n)}{(n+1)!} = \frac{n+1}{m-n}$$

Hence $\lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \left|\frac{n+1}{m-n}\right| = \lim \left|\frac{1+1/n}{1-m/n}\right| = 1$, or $l = 1$.

Similarly, $a_0 + a_1(x-a) + \cdots$ is called a power series in $x - a$, and $a_0 + a_1(1/x) + \cdots$ a power series in $1/x$. Evidently if $\Sigma a_n x^n$ converges when $|x| < l$, then $\Sigma a_n(x-a)^n$ converges when $|x-a| < l$, and $\Sigma a_n(1/x^n)$ converges when $|x| > l$.

Example 1. Show that the following series converge for the values of x indicated.

1. $3x + \frac{(3x)^2}{2} + \frac{(3x)^3}{3} + \cdots$ $\qquad |x| < 1/3$

2. $(x-2) + 2^2(x-2)^2 + 3^2(x-2)^3 + \cdots$ $\qquad 1 < x < 3$

3. $\frac{1}{3}x + \frac{1\cdot 3}{3\cdot 6}x^2 + \frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9}x^3 + \cdots$ $\qquad |x| < 3/2$

4. $mx + \frac{m(m+3)}{2!}x^2 + \frac{m(m+3)(m+6)}{3!}x^3 + \cdots$ $\qquad |x| < 1/3$

5. $\frac{1}{x} + \frac{2}{x^2} + \cdots + \frac{n}{x^n} + \cdots$ $\qquad |x| > 1$

6. $\cdots\left(\frac{x}{2}\right)^{-3}+\left(\frac{x}{2}\right)^{-2}+\left(\frac{x}{2}\right)^{-1}+1+\frac{x}{3}+\left(\frac{x}{3}\right)^{2}+\left(\frac{x}{3}\right)^{3}+\cdots, 2<|x|<3$

EXAMPLE 2. Show that $1+x+2!\,x^2+3!\,x^3\cdots$ converges when $x=0$ only.

180. Functions defined by power series. For each value of x within its interval of convergence $(-l, l)$, the series $a_0+a_1x+\cdots$(1) has a definite sum; it therefore represents a function of x within $(-l, l)$, § 14. This function $f(x)$ we call the sum of the series (1) and write

$$f(x)=a_0+a_1x+a_2x^2+\cdots \qquad |x|<l$$

Conversely, it is often advantageous to represent a *given* function of x by a power series in x or $x-a$; but such a representation is valid within $(-l, l)$ only.

Thus $1/(1-x)$ may be represented by $1+x+x^2+\cdots$ when $|x|<1$, but then only.

181. Continuity. *The function* $f(x)=a_0+a_1x+a_2x^2+\cdots$, $|x|<l$, *is continuous at every point* $x=x_1$ *such that* $|x_1|<l$.

If $S_n(x)$ be the sum of the first n terms of $a_0+a_1x+\cdots$, and $R_n(x)$ the remainder after the nth term,

$$f(x)-f(x_1)=[S_n(x)-S_n(x_1)]+R_n(x)-R_n(x_1)$$

It is to be proved that $\lim\limits_{x\to x_1}[f(x)-f(x_1)]=0$.

Let c be any number between $|x_1|$ and l, and R_n the remainder after n terms in the series $|a_0|+|a_1c|+|a_2c^2|+\cdots$. This series being convergent, if any positive number ϵ be assigned, we can find a value n' of n such that when $n \geqq n'$, then $R_n<\epsilon$. But $|R_n(x_1)|<R_n$; and if $|x|<c$, then also $|R_n(x)|<R_n$; hence when $n\geqq n'$ and $|x|<c$, we shall have $|R_n(x_1)|, |R_n(x)|<\epsilon$.

Again, since $S_{n'}(x)$ is a polynomial in x and is therefore continuous, we can find a positive number δ such that, when $|x-x_1|<\delta$, then $|S_{n'}(x)-S_{n'}(x_1)|<\epsilon$.

We thus make $|f(x)-f(x_1)|<3\,\epsilon$. We can take $3\,\epsilon$ as small as we please. Hence $\lim\limits_{x\to x_1}[f(x)-f(x_1)]=0$.

182. Integration. If $f(x) = a_0 + a_1x + a_2x^2 + \cdots$, $|x| < l$, and x_1 be such that $|x_1| < l$, we can find $\int_0^{x_1} f(x)dx$ by integrating the series $a_0 + a_1x + a_2x^2 + \cdots$ term by term.

For, expressing $f(x)$ in the form $a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + R_n(x)$, we have

$$\int_0^{x_1} f(x)dx = a_0x_1 + a_1\frac{x_1^2}{2} + \cdots + a_{n-1}\frac{x_1^n}{n} + \int_0^{x_1} R_n(x)dx$$

By § 181, 1., if any positive number ϵ be assigned we can find n' such that, when $n \geqq n'$, then for every value of x in the interval $(0, x_1)$ we shall have $|R_n(x)| < \epsilon$, and this makes $|\int_0^{x_1} R_n(x)dx| < \epsilon|x_1|$. Hence $\lim\limits_{n\to\infty} \int_0^{x_1} R_n(x)dx = 0$, and

$$\int_0^{x_1} f(x)dx = a_0x_1 + a_1\frac{x_1^2}{2} + \cdots + a_{n-1}\frac{x_1^n}{n} + \cdots \tag{1}$$

183. Differentiation. Given $f(x) = a_0 + a_1x + a_2x^2 + \cdots$, $|x| < l$. If $|x| < l$, we can find $f'(x)$ by differentiating $a_0 + a_1x + a_2x^2 + \cdots$ term by term.

For the series got by differentiating $a_0 + a_1x + a_2x^2 + \cdots$ term by term, namely $a_1 + 2a_2x + \cdots$, also converges[1] when $|x| < l$. Hence, by § 182 (1), when $|x| < l$,

$$\int_0^x [a_1 + 2a_2x + \cdots + na_nx^{n-1} + \cdots]\,dx$$
$$= a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \tag{2}$$

But equating the derivative of the second member of (2) to that of the first gives

$$\frac{d}{dx}[a_1x + a_2x^2 + \cdots + a_nx^n + \cdots]$$
$$= a_1 + 2a_2x + \cdots + na_nx^{n-1} + \cdots \tag{3}$$

Hence, for $|x| < l$, the derivative of $a_1x + a_2x^2 + \cdots$, and therefore that of $f(x) = a_0 + a_1x + a_2x^2 + \cdots$, can be got by differentiating term by term.

[1] See p. 220, Ex. 12. In case $|a_n/a_{n+1}| \to l$, the following proof suffices:

$$\text{If } \left|\frac{a_n}{a_{n+1}}\right| \to l, \text{ then } \left|\frac{na_n}{(n+1)a_{n+1}}\right| \to l \text{ since } \frac{n}{n+1} \to 1.$$

184. The logarithmic series. We have

$$\log(1+x) = \int_0^x \frac{dx}{1+x}$$

Also
$$\int_0^x \frac{dx}{1+x} = \int_0^x (1 - x + x^2 - \cdots)dx$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \quad |x| < 1$$

Hence for $|x| < 1$

$$\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots \qquad (1)$$

This series is not very serviceable[1] for computing logarithms; but

$$\log\frac{1+x}{1-x} = \log(1+x) - \log(1-x)$$
$$= 2\left[\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right] \qquad (2)$$

and from (2) by setting $\dfrac{1+x}{1-x} = \dfrac{n+1}{n}$, $\therefore\ x = \dfrac{1}{2n+1}$, we get

$$\log(n+1) = \log n + 2\left[\frac{1}{2n+1} + \frac{1}{3}\,\frac{1}{(2n+1)^3} + \cdots\right] \qquad (3)$$

Hence, since $\log 1 = 0$, we have

$$\log 2 = 2\left[\frac{1}{3} + \frac{1}{3^4} + \frac{1}{5}\,\frac{1}{3^5} + \cdots\right] = .6931\cdots;$$

$$\log 3 = \log 2 + 2\left[\frac{1}{5} + \frac{1}{3}\,\frac{1}{5^3} + \cdots\right] = 1.0986; \text{ and so on.}$$

EXAMPLE. Using $\tan^{-1}x = \displaystyle\int_0^x \frac{dx}{1+x^2}$, prove that

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots, \quad |x| < 1$$

185. The binomial series. Let $f(x)$ denote the sum of the binomial series, § 179, 3, so that, for $|x| < 1$,

$$f(x) = 1 + \frac{m}{1}x + \frac{m(m-1)}{2!}x^2 + \cdots \qquad (1)$$

We are to prove that $f(x) = (1+x)^m$.

[1] It is available only when $|x| < 1$, and even then it converges slowly.

Differentiating both members of (1), we obtain

$$f'(x) = m\left[1 + \frac{m-1}{1}x + \frac{(m-1)(m-2)}{2!}x^2 + \cdots\right] \quad (2)$$

But if to the series in brackets we add x times this series, combining terms in like powers of x, we obtain the series (1). Hence

$$f'(x)(1+x) = mf(x)$$

We therefore have successively,

$$\frac{f'(x)}{f(x)} = \frac{m}{1+x}, \qquad \int_0^x \frac{f'(x)}{f(x)}dx = m\int_0^x \frac{dx}{1+x},$$

$$\log f(x) = \log (1+x)^m, \qquad f(x) = (1+x)^m$$

Hence, for any real value of m,

$$(1+x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{2!}x^2 + \cdots$$
$$+ \frac{m(m-1)\cdots(m-n+1)}{n!}x^n + \cdots \qquad |x| < 1 \quad (3)$$

EXERCISE XXXVII

By aid of (3), show that

1. $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2 \cdot 4}x^2 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \cdots \qquad |x| < 1$

2. $\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots \qquad |x| < 1$

3. $\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \cdots \qquad |x| < 1$

4. Using $\sin^{-1} x = \int_0^x \frac{dx}{\sqrt{1-x^2}}$ and the series in Ex. 3, prove that

$$\sin^{-1} x = \frac{x}{1} + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots \qquad |x| < 1$$

5. Using $\sin^{-1}(1/2) = \pi/6$, show that $\pi = 3.14\cdots$

6. Show that replacing x by x/a in (3) gives, for $|x| < |a|$

$$(a+x)^m = a^m + \frac{m}{1}a^{m-1}x + \cdots + \frac{m(m-1)\cdots(m-n+1)}{n!}a^{m-n}x^n + \cdots$$

7. Compute $\sqrt{17} = (16+1)^{1/2} = 4(1 + 1/16)^{1/2}$ to the fourth decimal figure.

8. By aid of (3), show that $\int_0^1 (4 + x^3)^{1/2}dx = 2.0604$.

9. For $|x| > 1$, show that

$$\sqrt{x+1} = \sqrt{x}\left[1 + \frac{1}{2}\cdot\frac{1}{x} - \frac{1}{2\cdot 4}\frac{1}{x^2} + \frac{1\cdot 3}{2\cdot 4\cdot 6}\frac{1}{x^3} - \cdots\right]$$

10. Show that the difference between corresponding ordinates of the hyperbola $x^2 - y^2 = 1$ and its asymptote $y = x$ is approximately $-1/2\,x$.

11. Find to the third decimal figure the length of an arc of the parabola $y^2 = 4\,x$ between $x = 10$ and $x = 100$.

12. Show that when $0 < x < 1$, then $\sqrt{1+x} = 1 + \frac{x}{2}$ with an error which is $< \frac{x^2}{8}$, and $\sqrt{1-x} = 1 - \frac{x}{2}$ with an error which is $< \frac{x^2}{8(1-x)}$.

13. If the coefficients of $f(x) = a_0 + a_1x + \cdots$ from a_n on are numerically less than some positive number c, show that when $|x| < 1$, then $f(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ with an error which is numerically less than $cx^n/(1-x)$.

14. Compute log 4, log 5, log 6, each to the fourth decimal figure.

15. Express the coefficients of x^n in Exs. 1. and 2. in terms of factorials, and powers of 2.

16. If $f(x) = 1 + \frac{x}{1} + \cdots + \frac{x^n}{n!} + \cdots$, show successively that

$$f(x) = f'(x), \quad \log f(x) = x, \quad f(x) = e^x$$

186. Taylor's and Maclaurin's series. Under what conditions may a given function $f(x)$ be represented by a power series, and what is the general form of this series? The second question is readily answered, as follows.

Let $f(x)$ denote a given function of x, and a any assigned value of x. Assume that a power series of the form $A_0 + A_1(x-a) + A_2(x-a)^2 + \cdots$ exists which in its interval of convergence $(-l, l)$ represents $f(x)$, so that, for $|x - a| < l$,

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \cdots + A_n(x-a)^n + \cdots$$

Then, § 183, we also have, for $|x - a| < l$,

$$f'(x) = A_1 + 2\,A_2(x-a) + 3\,A_3(x-a)^2 + \cdots$$
$$f''(x) = 2\cdot 1\cdot A_2 + 3\cdot 2\cdot A_3(x-a) + \cdots$$

and so on; and by setting $x = a$ in these equations we find $A_0 = f(a)$, $A_1 = f'(a)$, $A_2 = f''(a)/2!, \cdots, A_n = f^{(n)}(a)/n! \cdots$

Hence, if $f(x)$ admits of being expressed as a power series in $x - a$, the expression is

$$f(x) = f(a) + f'(a)\frac{x-a}{1} + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} + \cdots \tag{1}$$

The series (1) is called *Taylor's series.* When a is 0, it becomes

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + \cdots \tag{2}$$

which is called *Maclaurin's series.*

EXAMPLE 1. If $\sin x$ can be represented by a power series in x, what is the series?

$$f(x) = \sin x \quad f'(x) = \cos x \quad f''(x) = -\sin x \quad f'''(x) = -\cos x, \ldots$$

$$\therefore f(0) = 0 \quad f'(0) = 1 \quad f''(0) = 0 \quad f'''(0) = -1, \ldots$$

Hence $$\sin x = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

EXAMPLE 2. Assuming that the following functions have Maclaurin series, show that

1. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$

2. $e^x = 1 + \dfrac{x}{1} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

3. $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \cdots$

4. $\sec x = 1 + \dfrac{x^2}{2} + \dfrac{5}{24}x^4 + \cdots$

EXAMPLE 3. Show by (1) that

$$x^3 = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3.$$

EXAMPLE 4. Verify the fact that the series already got for $\log(1 + x)$ and $(1 + x)^m$ are Maclaurin series.

EXAMPLE 5. Show that $f(a + x) = f(a) + f'(a)\dfrac{x}{1} + f''(a)\dfrac{x^2}{2!} + \cdots$

187. An extension of the mean value theorem. To find under what conditions a given function $f(x)$ is expressible by a Taylor series, we extend the mean value theorem (§ 97) as follows:

Suppose that $f(x)$ has a finite nth derivative throughout the interval (a, b), and therefore that $f^{(n-1)}(x), \cdots, f'(x), f(x)$ are continuous in (a, b).

Let K denote the constant determined by the equation:

$$f(b) - f(a) - f'(a)\frac{b-a}{1} - f''(a)\frac{(b-a)^2}{2!} - \cdots - f^{(n-1)}(a)\frac{(b-a)^{n-1}}{(n-1)!} - K\frac{(b-a)^n}{n!} = 0 \qquad (1)$$

If we replace a by x in the first member of (1), we get a function

$$F(x) = f(b) - f(x) - f'(x)\frac{b-x}{1} - \cdots - f^{(n-1)}(x)\frac{(b-x)^{n-1}}{(n-1)!} - K\frac{(b-x)^n}{n!} \qquad (2)$$

which satisfies the conditions of Rolle's theorem, § 96; for $F(a) = 0$ by (1), $F(b) = 0$ identically, and $F'(x)$, which reduces to

$$F'(x) = -f^{(n)}(x)\frac{(b-x)^{n-1}}{(n-1)!} + K\frac{(b-x)^{n-1}}{(n-1)!} \qquad (3)$$

is finite in (a, b).

Hence there exists a value x_1 of x between and distinct from a and b, such that $F'(x_1) = 0$, and therefore, by (3), such that

$$-f^{(n)}(x_1)\frac{(b-x_1)^{n-1}}{(n-1)!} + K\frac{(b-x_1)^{n-1}}{(n-1)!} = 0, \text{ or } K = f^{(n)}(x_1) \qquad (4)$$

Substituting this value of K in (1) and transposing terms, we obtain

$$f(b) = f(a) + f'(a)\frac{b-a}{1} + \cdots + f^{(n-1)}(a)\frac{(b-a)^{n-1}}{(n-1)!} + f^{(n)}(x_1)\frac{(b-a)^n}{n!} \qquad (5)$$

where x_1 is a definite number between a and b and may be written

$$x_1 = a + \theta(b - a) \text{ where } 0 < \theta < 1 \tag{6}$$

188. Existence of a Taylor series. Let x denote a variable in (a, b). In (5) replace b by x. We get

$$f(x) = f(a) + f'(a)\frac{x-a}{1} + \cdots + f^{(n-1)}(a)\frac{(x-a)^{n-1}}{(n-1)!} + R_n \tag{7}$$

where

$$R_n = f^{(n)}[a + \theta(x - a)]\frac{(x-a)^n}{n!}$$

We call (7) *Taylor's series with remainder term.*

Suppose that the condition that $f^{(n)}(x)$ is finite in (a, b) holds good for all finite values of n, and that $R_n \to 0$ when $n \to \infty$. We then have

$$f(x) = f(a) + f'(a)\frac{x-a}{1} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} + \cdots \tag{8}$$

The Taylor series (8) *converges and represents* $f(x)$ *in* (a, b) *if all the derivatives of* $f(x)$ *are finite in* (a, b) *and if* $\lim\limits_{n\to\infty} R_n = 0$.

In the case of Maclaurin's series, a is 0 and (7) becomes

$$f(x) = f(0) + f'(0)\frac{x}{1} + \cdots + f^{(n-1)}(0)\frac{x^{n-1}}{(n-1)!} + R_n$$

where

$$R_n = f^{(n)}(\theta x)\frac{x^n}{n!} \tag{9}$$

If in $(0, b)$ *every* $f^{(n)}(x)$ *is finite and* $\lim\limits_{n\to\infty} R_n = 0$, *then in* $(0, b)$

$$f(x) = f(0) + f'(0)\frac{x}{1} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + \cdots \tag{10}$$

189. The function e^x. Here $f^{(n)}(x) = e^x$ is finite for every finite n and x.

$$f^{(n)}(\theta x)\frac{x^n}{n!} = e^{\theta x}\frac{x^n}{n!}$$

which $\to 0$ when $n \to \infty$; for $e^{\theta x}$ is finite, and since $x^n/n!$ is the nth term of a convergent series, § 179, $\lim\limits_{n\to\infty} x^n/n! = 0$. Hence, by (10),

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad |x| < \infty \tag{11}$$

190. The functions sin x, cos x. If $f(x) = \sin x$, then $f'(x) = \cos x = \sin (x + \pi/2)$, and by repetitions of this process $f^{(n)}(x) = \sin \left(x + n\frac{\pi}{2}\right)$. Hence

$$f^{(n)}(\theta x)\frac{x^n}{n!} = \sin\left(\theta x + n\frac{\pi}{2}\right)\frac{x^n}{n!}$$

which $\to 0$ when $n \to \infty$, if $|x| < \infty$. Similarly for $\cos x$. Hence by (10)

$$\sin x = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad |x| < \infty \qquad (12)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad |x| < \infty \qquad (13)$$

Here x represents the circular measure of the angle: $1° = \pi/180$ or .017453 radians.[1]

Example 1. Compute sin 10° to the fourth decimal figure.

Since $10° = .17453$ radians, and (12) is an alternating series, we have, by the remark at the end of § 178,

$$\sin 10° = .17453 - \frac{(.17453)^3}{6}, \text{ with error} < \frac{(.17453)^5}{120}, \quad \therefore < (.1)^5.$$

Hence $\qquad \sin 10° = .17453 - .00089 = .1736 \cdots.$

Example 2. For how large an angle may we set $\sin x = x$ with an error $< .001$?

Example 3. For an arc which subtends the angle θ at the center of a circle of radius r show that the difference between the arc and its chord is $r[\theta - 2\sin(\theta/2)]$.

Example 4. Calling the radius of the earth 4000 m., show of two points whose distance along the earth's surface is 100 m. that their distance in a straight line is about 13.75 ft. less than this.

Example 5. Compute cos 20° to the fourth decimal figure.

[1] In the case of $\log (1 + x)$ and of $(1 + x)^m$, and $0 < x < -1$, it will be found that we cannot prove that $R_n \to 0$ if we express R_n in the form $f^{(n)}(\theta x)x^n/n!$ (Lagrange's form). But if we replace the last term in § 187 (1) by $K(b - a)$, the reasoning in §§ 187, 188 will lead to the following expression for R_n due to Cauchy:

$$R_n = f^{(n)}(\theta x)(1 - \theta)^{n-1}\frac{x^n}{(n-1)!} \qquad 0 < \theta < 1$$

and for $\log (1 + x)$ and $(1 + x)^m$, and $|x| < 1$, this $R_n \to 0$.

EXAMPLE 6. For a circle of 4000 m. radius show that an arc 1 m. long differs from its chord less than .0002 in. and recedes from the chord less than 2 in.

EXAMPLE 7. Show that $\int_0^1 \frac{\sin x}{x} dx = \frac{17}{18}$, with an error $< \frac{1}{600}$.

EXAMPLE 8. Show by (11) that $e = 2.71828 \ldots$.

EXAMPLE 9. From (11), (12), (13) derive the expansions of the following in powers of x:

1. $\sqrt{e^x} = e^{x/2}$ 2. $1/e^x$ 3. $\cos^2 x = (1 + \cos 2x)/2$ 4. $\sin x \cos x$

191. Operations with power series. The series for e^x, $\sin x$, $\cos x$, $\log (1 + x)$ and $(1 + x)^m$ are very important and should be memorized. For other functions $f(x)$ than these the direct derivation of the series $f(0) + f'(0)x + \cdots$ is, generally speaking, difficult, and the proof of its equivalence to $f(x)$ — by showing that $\lim R_n = 0$ — impracticable. But for a function which is expressible in terms of e^x, $\sin x$, $\cdots$ this may often be accomplished indirectly by aid of the following theorems. For, by § 186, a power series in x, however obtained, which in its interval of convergence equals $f(x)$, is the Maclaurin series of $f(x)$.

1. The rules for adding and multiplying polynomials in x may be extended as follows to functions defined by power series in x.

If when $|x| < l$ *we have*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$
$$\phi(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$$

then also, for $|x| < l$,

$$f(x) + \phi(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$$
$$f(x)\phi(x) = a_0 b_0 + (a_1 b_0 + a_0 b_1)x + (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 + \cdots$$

For if S_n and S'_n be the sums of the first n terms of the $f(x)$ and $\phi(x)$ series, we have

$$\lim S_n + \lim S'_n = \lim (S_n + S'_n)$$

which proves the theorem for $f(x) + \phi(x)$. The proof for $f(x)\phi(x)$ will be given later.

EXAMPLE 1. Find to four terms the Maclaurin series for the product $(1+x)^{-3}(1+2\,x)^{-1}$.

$$\begin{array}{rlllll}
\text{By § 185 (3), } (1+x)^{-3} = & 1 - 3\,x & + 6\,x^2 & - 10\,x^3 & + \cdots & |x|<1 \\
(1+2\,x)^{-1} = & 1 - 2\,x & + 4\,x^2 & - \;\; 8\,x^3 & + \cdots & |x|<1/2 \\
\hline
(1+x)^{-3}(1+2\,x)^{-1} = & 1 - 3\,|\,x & + 6\,|\,x^2 & - 10\,|\,x^3 & + \cdots & \\
 & \quad\; - 2\,| & + 6\,| & - 12\,| & & \\
 & & + 4\,| & - 12\,| & & \\
 & & & - \;\; 8\,| & & \\
\hline
= & 1 - 5\,x & + 16\,x^2 & - 42\,x^3 & + \cdots & |x|<1/2
\end{array}$$

This is the Maclaurin series of $f(x)=(1+x)^{-3}(1+2\,x)^{-1}$ to four terms, that is, $f(0)=1$, $f'(0)=-5$, $f''(0)=32$, $f'''(0)=-252$. Observe also that the reckoning not only gives the series but also proves its equivalence to $f(x)$ for $|x|<1/2$.

2. By the process of long division, or the method of undetermined coefficients, a rational fraction

$$\frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_mx^m},\quad b_0\neq 0,$$

may be transformed into an infinite series which converges and represents the fraction when $|x|$ is less than the numerical value of the numerically least root, real or imaginary, of $b_0+b_1x+\cdots+b_mx^m=0$. The like is true of the fraction $f(x)/\phi(x)$ when $f(x)$ and $\phi(x)$ are infinite series, the resulting series then being convergent and representing $f(x)/\phi(x)$ when $|x|$ is less than the limit of convergence of $f(x)$ and also less than the numerical value of the numerically least root of $\phi(x)=0$. This and the theorem in 3 will be proved later.

EXAMPLE 1. Express $(2-3\,x)/(1-3\,x+2\,x^2)$ by a Maclaurin series.

Let
$$\frac{2-3\,x}{1-3\,x+2\,x^2}=C_0+C_1x+C_2x^2+C_3x^3+\cdots$$

Clearing of fractions,

$$\begin{array}{rllll}
2-3\,x = & C_0 + C_1\,|\,x & + \;\; C_2\,|\,x^2 & + \;\; C_3\,|\,x^3 & + \cdots \\
 & \quad\; - 3\,C_0\,| & - 3\,C_1\,| & - 3\,C_2\,| & \\
 & & + 2\,C_0\,| & + 2\,C_1\,| &
\end{array}$$

Hence $C_0=2$, $C_1-3\,C_0=-3$, $C_2-3\,C_1+2\,C_0=0$, $\cdots$

Therefore $C_0=2$, $C_1=3$, $C_2=5$, $C_3=9$, $\cdots$

and the required series, to four terms, is $2 + 3x + 5x^2 + 9x^3 + \cdots$. The roots of $1 - 3x + 2x^2 = 0$ are $1/2$ and 1; hence the series converges when $|x| < 1/2$. This may be proved by resolving the given fraction into its partial fractions:

$$\frac{2 - 3x}{1 - 3x + 2x^2} = \frac{1}{1 - x} + \frac{1}{1 - 2x} = (1 + x + x^2 + \cdots)$$
$$+(1 + 2x + 4x^2 + \cdots) = 2 + 3x + 5x^2 + \cdots, \quad |x| < 1/2$$

EXAMPLE 2. Find to three terms the Maclaurin series for $\tan x$.

$$\begin{array}{rl|ll}
\sin x = & x - x^3/6 + x^5/120 - \cdots & 1 - x^2/2 + x^4/24 + \cdots & = \cos x \\
 & x - x^3/2 + x^5/24 \;\; - \cdots & x + x^3/3 + 2x^5/15 + \cdots & = \tan x \\
\hline
 & x^3/3 - x^5/30 \;\; + \cdots & & \\
 & x^3/3 - x^5/6 \;\;\; + \cdots & & \\
\hline
 & 2x^5/15 \;\; - \cdots & &
\end{array}$$

The series converges when $|x| < \pi/2$, one of the numerically smallest roots of $\cos x = 0$.

3. *If in the series* $\quad z = a_0 + a_1y + a_2y^2 + \cdots \qquad |y| < l$
we substitute $\quad y = b_0 + b_1x + b_2x^2 + \cdots \qquad |b_0| < l$
and then compute $y^2, y^3, \cdots$ *by* 1, *and finally collect terms which involve like powers of* x, *we obtain a power series in* x *which converges and represents* z, *at least when* $|b_0| + |b_1x| + \cdots < l$.

EXAMPLE. Find the Maclaurin series for $(\cos x)^{1/3}$ to the term in x^4.

$$\begin{aligned}
(\cos x)^{1/3} &= \left[1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\right]^{1/3} \\
&= 1 + \frac{1}{3}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - \frac{1}{9}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^2 \\
&= 1 - \frac{x^2}{6} - \frac{x^4}{72} - \cdots.
\end{aligned}$$

4. All that has been said in this section applies also to the representation of a function in powers of $x - a$. In the case of a rational fraction $F(x)$ in particular, often the Taylor series in $x - a$ is most easily got by setting $x = a + (x - a)$ in $F(x)$ and then applying the theorem in 2. The expression of this $F(x)$ in process of $1/x$ may also be got by first expressing $F(x)$ in terms of $1/x$ and then applying 2.

Thus the expansions of $1/(x+1)$ in powers of $x-1$ and of $1/x$ may be got as follows:

$$\frac{1}{x+1}=\frac{1}{2+(x-1)}=\frac{1}{2}\,\frac{1}{1+(x-1)/2}$$

$$=\frac{1}{2}\left[1-\frac{x-1}{2}+\frac{(x-1)^2}{4}-\cdots\right] \qquad |x-1|<2$$

$$\frac{1}{x+1}=\frac{1}{x}\left[\frac{1}{1+1/x}\right]=\frac{1}{x}\left[1-\frac{1}{x}+\frac{1}{x^2}-\cdots\right]=\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\cdots$$

$$|x|>1$$

192. Indeterminate forms. Infinite series may often be used to advantage in evaluating indeterminate forms.

EXAMPLE 1. $F(x)=(\sin x-x)/x^2\sin x$ becomes $0/0$ when $x=0$. Find $\lim\limits_{x\to 0} F(x)$.

$$\lim_{x\to 0}\frac{\sin x-x}{x^2\sin x}=\lim_{x\to 0}\frac{-x^3/3!+x^5/5!-\cdots}{x^3-x^5/3!+\cdots}$$

$$=\lim_{x\to 0}\frac{-1/3!+x^2/5!\cdots}{1-x^2/3!+\cdots}=-\frac{1}{6}$$

EXAMPLE 2. Prove that $\lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n=e^x$

$$\log\left(1+\frac{x}{n}\right)^n=n\log\left(1+\frac{x}{n}\right)=n\left(\frac{x}{n}-\frac{1}{2}\frac{x^2}{n^2}+\cdots\right)\to x$$

Hence
$$\left(1+\frac{x}{n}\right)^n\to e^x$$

EXERCISE XXXVIII

1. Find the following limiting values:

1. $\lim\limits_{x\to 0}\dfrac{2x\cos x-\sin 2x}{x^2\log(1+x)}$

2. $\lim\limits_{x\to 0}\dfrac{(1+x)-(1+2x)^{1/2}}{(1+3x)^{1/3}-(1+4x)^{1/4}}$

3. $\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}-2}{\tan^2 x}$

4. $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}\right)^{bx}$

2. Using $\sec x=1/\cos x$, prove that

$$\sec x=1+\frac{x^2}{2}+\frac{5x^4}{24}+\cdots,\ |x|<\frac{\pi}{2}.$$

3. In leveling, the curvature of the earth is ordinarily neglected. Show that the consequent error for points 1 m. apart is about 8 in.

4. Show that the difference between the perimeter of the circumscribed and inscribed regular polygons of n sides to a circle of radius r is $r\left(\dfrac{\pi^3}{n^2}+\dfrac{\pi^5}{4n^4}+\cdots\right)$.

5. Referring to § 73, show that for $|x| < \infty$

$$\sinh x = \frac{x}{1} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

6. Show that $\log (x+3)^{1/2} = \frac{1}{2}\left(\log 3 + \frac{x}{3} - \frac{1}{2}\frac{x^2}{9} + \cdots\right) \qquad |x| < 3$

7. Find the Maclaurin series to the term in x^4 and its limit of convergence for

1. $\dfrac{2-x}{(2+x)(1+x^2)}$ 2. $(1 - 2x - x^2)^{1/2}$ 3. $e^{\sin x}$

8. Show that $\dfrac{x^2}{x^2 - 3x + 2} = 1 + \dfrac{3}{x} + \dfrac{7}{x^2} + \dfrac{15}{x^3} + \cdots$ for $|x| > 2$.

193. Taylor series with remainder term. Approximations. 1. Suppose that $f(x)$ has a finite nth derivative in the interval $(a, a+h)$. Then by setting $x = a + h$ in the formula § 188 (7) we have

$$f(a+h) = f(a) + f'(a)\frac{h}{1} + \cdots + f^{(n-1)}(a)\frac{h^{n-1}}{(n-1)!} + f^{n}(a + \theta h)\frac{h^n}{n!} \qquad (1)$$

If M_n denote the greatest value of $|f^{(n)}(x)|$ in $(a, a+h)$, the last term in (1) is numerically $\leqq M_n h^n/n!$ and can therefore, by taking h sufficiently small, be made as small as we please. Hence, for small values of h,

$$f(a+h) = f(a) + f'(a)\frac{h}{1} + \cdots + f^{(n-1)}(a)\frac{h^{n-1}}{(n-1)!} \qquad (2)$$

approximately, with an error numerically $\leqq M_n h^n/n!$

We have $a + h = x$; hence when a is 0, then h is x and (1) is the Maclaurin expansion, with remainder term, of $f(x)$.

EXAMPLE 1. The Maclaurin expansion of $(1-x)^{1/2}$ to three terms is $1 - \frac{x}{2} - \frac{x^2}{8}$. How great is the error involved in setting

$$(1-x)^{1/2} = 1 - \frac{x}{2} - \frac{x^2}{8} \text{ when } 0 < x \leqq 1/4?$$

$$f'''(x) = -\tfrac{3}{8}(1-x)^{-5/2} \quad \therefore M_3 = \tfrac{3}{8}(\tfrac{4}{3})^{5/2} = .7698.$$

$$\therefore \text{error is} < \frac{.7698}{6 \cdot 4^3} = .002005$$

2. In (1) transpose the term $f(a)$, then replace a by x and h by Δx; we get

$$\Delta f(x) = f'(x)\Delta x + f''(x)\frac{(\Delta x)^2}{2!} + \cdots + f^{(n-1)}(x)\frac{(\Delta x)^{n-1}}{(n-1)!}$$

with an error E numerically less than $M_n(\Delta x)^n/n!$, M_n denoting the greatest value of $|f^{(n)}(x)|$ in $(x, x+\Delta x)$. Thus

$$\Delta f(x) = f'(x)\Delta x, \quad |E| \leqq M_2\frac{(\Delta x)^2}{2!} \tag{3}$$

$$\Delta f(x) = f'(x)\Delta x + f''(x)\frac{(\Delta x)^2}{2!}, \quad |E| \leqq M_3\frac{(\Delta x)^3}{3!} \tag{4}$$

and so on.

EXAMPLE 2. If $f(x) = 1/x$, and x and Δx are positive, the greatest value of $f''(x) = 2/x^3$ in $(x, x+\Delta x)$, is $2/x^3$, and therefore, by (3),

$$\Delta\left(\frac{1}{x}\right) = -\frac{1}{x^2}\Delta x, \text{ with error } < \frac{(\Delta x)^2}{x^3}$$

For what values of x will this formula give $\Delta(1/x)$ correctly to the fifth decimal figure when $\Delta x = .1$?

We must have $\dfrac{(.1)^2}{x^3} < .00001 \quad \therefore x^3 > 1000 \quad \therefore x > 10$

Hence for $x > 10$, and $0 < \Delta x \leqq .1$, we have

$$\frac{1}{x+\Delta x} = \frac{1}{x} - \frac{\Delta x}{x^2}, \text{ with error } < .00001.$$

Show that for $\Delta x = .1$ the formula (4) gives

$$\Delta\left(\frac{1}{x}\right) \text{ with an error } < .00001 \text{ when } x > 5.63.$$

194. Rule of proportional parts. Suppose that, as in the case of $\log_{10} x$ and $\sin x$, the function $f(x)$ is tabulated for a sequence of values of x having some common difference h. It is customary to compute $f(x)$ for intermediate values of x by a rule which assumes that for small changes in x the change in $f(x)$ is proportional to that in x: so that if a be a number in the table, $a+h$ the next number, and $a+th$ $(0 < t < 1)$ a given intermediate number, then

$$\frac{f(a+th) - f(a)}{f(a+h) - f(a)} = \frac{th}{h} = \frac{t}{1} \tag{1}$$

By how much may the value of $f(a + th)$ given by (1) be in error? By (1) and § 193 (1),

$$f(a + th) = f(a) + t\,[f(a + h) - f(a)]$$
$$= f(a) + th\,f'(a) + \frac{th^2}{2} f''(x_1) \qquad a < x_1 < a + h \qquad (2)$$

And, by § 193 (1), the actual value of $f(a + th)$ may be expressed by

$$f(a + th) = f(a) + th\,f'(a) + \frac{t^2h^2}{2} f''(x_2) \qquad a < x_2 < a + th \quad (3)$$

Hence the error in question is the difference between (2) and (3), or $[f''(x_1)t - f''(x_2)t^2]\,h^2/2$, which is obviously less than M_2h^2, and can be proved to be less than $M_2h^2/8$, M_2 denoting the greatest value of $|f''(x)|$ in $(a, a + h)$.

Example 1. In a seven-place logarithmic table the logarithms of all numbers from 10000 to 100000 are given at unit intervals. Here $h = 1$ and

$$f(x) = \log_{10} x \quad \therefore f''(x) = -\frac{\mu}{x^2} \quad \text{where } \mu = \log_{10} e = .43429 \cdots$$

Hence the greatest value of $|f''(x)|$ for $x \geqq 10000$ is $.43429/(10000)^2$, and the error resulting from the use of the method of proportional parts at any point in the table is less than $.43429/8(10000)^2 = .000000000543$ or far less than a unit in the seventh place.

Example 2. Find the corresponding error in the case of a four-place table.

Example 3. Let A and B be the points of the graph of $y = f(x)$ corresponding to $x = a$ and $x = a + h$, and P and Q the points of the arc AB and the chord AB corresponding to $x = a + th$. Show that the length of PQ cannot exceed $M_2h^2/8$.

195. Area under a parabolic arc. The prismoid formula. 1. Let y_a, y_c, y_b be any positive ordinates erected at points a, c, b of Ox such that $c - a = b - c = h$, and C_a, C_c, C_b their end points. There is a single parabola with axis parallel to Oy through the

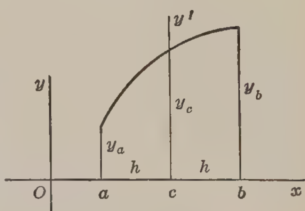

Fig. 109.

points C_a, C_c, C_b, and the area T of the space bounded by this parabola and Ox, $x = a$, $x = b$ is

$$T = \frac{h}{3}(y_a + 4\,y_c + y_b) \tag{1}$$

For let the equation of the parabola when c is taken as origin be $y = A + Bx' + Cx'^2$. Then

$$T = \int_{-h}^{h}(A + Bx' + Cx'^2)dx' = 2\,Ah + \tfrac{2}{3}\,Ch^3 \tag{2}$$

$$y_a = A - Bh + Ch^2 \qquad y_c = A \qquad y_b = A + Bh + Ch^2 \tag{3}$$

If we solve the equations (3) for A and C in terms of y_a, y_c, y_b, and h, and substitute the results in (2), we get (1).

2. Let $y = f(x)$ be *any* function whose graph passes through C_a, C_c, C_b and which has a finite fourth derivative in (a, b). The area $S = \int_a^b f(x)dx$ of the space bounded by $y = f(x)$ and Ox, $x = a$, $x = b$ differs numerically from T by less than $M_4h^5/90$, where M_4 denotes the greatest value of $|f^{\text{iv}}(x)|$ in (a, b).

For,[1] setting $a = c - h$, $b = c + h$ in (1), we find

$$T = \frac{h}{3}[f(c - h) + 4f(c) + f(c + h)]$$

and if $F(x)$ denote an integral of $f(x)$, then

$$S = F(c + h) - F(c - h)$$

Let $\phi(h) = T - S$. It will be found that $\phi(h)$, $\phi'(h)$, $\phi''(h)$ vanish when $h = 0$ and that

$$\phi'''(h) = \frac{h}{3}[f'''(c + h) - f'''(c - h)]$$

which by the mean value theorem, § 97, can be reduced to the form

$$\phi'''(h) = \frac{2\,h^2}{3}f^{\text{iv}}(x_1) \qquad c - h < x_1 < c + h$$

Since $\phi''(0) = 0$ and $|f^{\text{iv}}(x_1)| < M_4$, we have, by § 129, Ex. 7,

$$|\phi''(h)| = \left|\int_0^h \phi'''(x)dx\right| < M_4\int_0^h \frac{2\,x^2}{3}dx = \frac{2}{3\cdot 3}M_4h^3$$

and repetitions of this process give

$$|\phi(h)| < \frac{2}{3\cdot 3\cdot 4\cdot 5}M_4h^5 = \frac{M_4h^5}{90}$$

[1] Ch.-J. de la Vallée Poussin, Cours d'Analyse Infinitésimale.

3. When $f(x)$ is a polynomial in x of degree less than four, we have $f^{\text{iv}}(x) \equiv 0$, hence $S = T$, and therefore, by (1),

$$\int_a^b f(x)dx = \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\frac{b-a}{6} \tag{8}$$

By § 141, if the area of a variable section of a solid by a plane perpendicular to Ox be $f(x)$, the volume V of the solid between the planes $x = a$ and $x = b$ is $\int_a^b f(x)dx$. Hence, when $f(x)$ is a polynomial in x of degree less than four,

$$V = \frac{d}{6}(B_1 + 4B_2 + B_3) \tag{9}$$

where d is the distance between the end sections, B_1, B_3 the areas of these sections, and B_2 that of the midsection. This is called the *prismoid formula*.[1] All the volume formulas of elementary geometry follow from it.

EXAMPLE. For the sphere got by revolving the circle $x^2 + y^2 = a^2$ about Ox, we have $f(x) = (a^2 - x^2)\pi$; also

$$B_1 = 0,\ B_3 = 0,\ B_2 = \pi a^2 \quad \therefore \text{ by (9), } V = \frac{2a}{6}(4\pi a^2) = \frac{4}{3}\pi a^3.$$

196. Simpson's rule. A number of methods have been devised for finding approximate values of a definite integral $\int_a^b f(x)dx$ when $\int f(x)dx$ cannot be got by the formulas of integration and when the Taylor series for $f(x)$ in powers of $x - a$ converges too slowly to be serviceable. One of the most useful is the following, deduced from the formula § 195 (1) and called *Simpson's rule.*

To find an approximate value of $S = \int_a^b f(x)dx$ *when* $f(x)$ *is positive and has a finite fourth derivative in* (a, b), *divide* (a, b) *into* $2n$ *equal parts of length* $h = (b - a)/2n$. *Let* x_1, x_2, $\cdots x_{2n-1}$ *denote the points of division. Compute the corre-*

[1] A prismoid is a solid whose bases B_1, B_3 are polygons in parallel planes and whose lateral surface can be divided into triangles whose bases and vertices are sides and angular points of B_1 and B_3. One can show for any prismoid that if Ox be taken perpendicular to B_1, B_3 the area $f(x)$ of a variable section parallel to B_1, B_3 is of the 2nd, 1st, or 0 degree in x, therefore that (9) holds good.

sponding ordinates $y_1, y_2, \cdots y_{2n-1}$, *also* y_a *and* y_b, *and substitute in the formula:*

$$T = \frac{h}{3}\{y_a + y_b + 4(y_1 + y_3 + \cdots) + 2(y_2 + y_4 + \cdots)\} \quad (1)$$

The numerical difference between S *and* T *is less than* $\dfrac{M(b-a)}{180}h^4$, *M being the greatest value of* $|f^{\text{iv}}(x)|$ *in* (a, b).

For let $C_a, C_1, C_2, \cdots C_b$ be the end points of the ordinates $y_a, y_1, y_2, \cdots y_b$. The right member of (1) is the sum of the areas under the parabolic arcs $C_aC_1C_2$, $C_2C_3C_4$, and so on, as given by § 195 (1). There are n of these areas and each differs from the corresponding part of S by less than $Mh^5/90$. Therefore, since $b - a = 2\,nh$, T differs from S by less than $M(b-a)h^4/180$.

EXAMPLE 1. Find the value of π from the formula $\dfrac{\pi}{4} = \displaystyle\int_0^1 \frac{dx}{1+x^2}$.

If we divide (0, 1) into four equal parts ($h = 1/4$) at $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 3/4$, we find

$$y_a = 1,\ y_1 = .941,\ y_2 = .8,\ y_3 = .64,\ y_b = .5$$

Hence, by (1),

$$\frac{\pi}{4} = \frac{1}{4\cdot 3}[1.5 + 4(.941 + .64) + 2(.8)] = \frac{1}{4\cdot 3}(9.424) = \frac{1}{4}(3.14133)\ \ldots$$

which gives $\pi = 3.14133$, while the actual value is $3.14159 \ldots$. If we divide (0, 1) into ten equal parts ($h = .1$) we get $\pi = 3.141593$, which is correct to the last figure.

EXAMPLE 2. Find log 2 from the formula $\log 2 = \displaystyle\int_0^1 \frac{dx}{1+x}$ by Simpson's rule, dividing (0, 1) into four equal parts. Using the test number $\dfrac{M(b-a)}{180}h^4$, show that the result should be correct to the third decimal figure.

EXERCISE XXXIX

1. Evaluate $\int_0^1 (1 + x^3)^{1/2}\,dx$ by Simpson's rule ($h = 1/4$).

2. Show by integration of series that $\int_0^1 e^{-x^2}dx = .7467$.

3. By the prismoid formula, show that the volume of a spherical segment of height h is $h^2\pi(a - h/3)$, a being the radius of the sphere.

4. For what values of Δx will the formula $\Delta \sin x = \cos x \, \Delta x$ give results which are correct to the second decimal figure when $x = \pi/3$?

5. Two curves $y = f(x)$ and $y = \phi(x)$ are said to have contact of order n at $x = a$ if $f(a) = \phi(a)$, $f'(a) = \phi'(a)$, ..., $f^{(n)}(a) = \phi^{(n)}(a)$, but $f^{(n+1)}(a) \neq \phi^{(n+1)}(a)$. Show that the curves also cross at $x = a$ when n is even, but not when n is odd.

6. Find the curve $y = A + Bx + Cx^2 + Dx^3$ which has contact of the third order with the ellipse $x^2 - xy + y^2 = 1$ at the point (1, 1).

7. Prove Huygen's formula: the length of a circular arc is approximately $2B + \frac{1}{3}(2B - A)$ where A is the chord and B the chord of half the arc.

8. Show that the greatest distance from the shore at which a mast head 70 ft. above the water level would be visible is about 10.3 m.

9. Verify the formula $4 \tan^{-1}(1/5) - \tan^{-1}(1/239) = \pi/4$, and by its aid and the series $\tan^{-1} x = x - x^3/3 + \dots$, compute π to the sixth decimal figure.

10. By Example 2, p. 85, the differential of arc of the ellipse $x = a\cos\phi$, $y = b\sin\phi$ is $ds = a(1 - e^2\cos^2\phi)^{1/2}d\phi$ where $e = (a^2 - b^2)^{1/2}/a$ is less than 1. Show that the length of a quadrant of the ellipse is

$$\frac{\pi}{2} a \left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{e^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{e^6}{5} - \cdots\right]$$

11. Verify the fact that the formula § 195 (8) gives the correct value of the area under the curve $y = x^3 + 2x^2$ between $x = 1$ and $x = 3$.

12. Supply the details of the following general proof of the theorem (assumed in § 183) that, if l be the limit of convergence of $\Sigma a_n x^n$, then $\Sigma n a_n x^{n-1}$ also converges when $|x| < l$.

Suppose $|x| < l$, and let c be any number such that $|x| < c < l$. Since $\Sigma |a_n c^n|$ is convergent, every $|a_n c^n|$ is less than some finite number M and therefore

$$|n a_n x^{n-1}| = n \frac{|a_n c^n|}{c} \left(\frac{|x|}{c}\right)^{n-1} < n \frac{M}{c} \left(\frac{|x|}{c}\right)^{n-1}$$

The series $\Sigma n \dfrac{M}{c}\left(\dfrac{|x|}{c}\right)^{n-1}$ converges since $\dfrac{u_{n+1}}{u_n} \to \dfrac{|x|}{c}$, which is < 1.

Hence $\Sigma |n a_n x^{n-1}|$ converges when $|x| < c$ $\therefore$ when $|x| < l$.

XXI. DETERMINANTS

197. On permutations. When considering the permutations of a set of objects taken all at a time, we may choose some particular order of the objects as the normal order. Any given permutation is then said to have as many *inversions* as it presents instances in which an object precedes one which in the normal order follows it.

Thus, if the objects in normal order are 1, 2, 3, 4, the permutation 3241 has the four inversions 32, 31, 21, 41.

If two of the objects in a permutation are interchanged, the number of inversions is changed by an odd number.

For obviously when the objects are adjacent, there is a change of one in the number of inversions; and an interchange of two separated objects can be effected by an odd number of interchanges of adjacent objects.

198. Determinants. Take any *square array* of numbers, as

$$a_1 \qquad \begin{matrix} a_1 & a_2 \\ b_1 & b_2 \end{matrix} \qquad \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \qquad \text{and so on,}$$

where the letter indicates the row and the subscript the column in which each number or *element* of the array occurs.

With the elements of any such array, form all the products that can be formed by taking one factor and but one from each row and column. In each product arrange the factors so that the row marks (the letters) are in normal order, and then count the inversions of the column marks (the subscripts). When their number is even (or 0), give the product

the plus sign; when odd, the minus sign. The algebraic sum of all these signed products is called the *determinant* of the array and is represented by the array itself with a bar placed at either side of it.

The number of rows or columns in the array is called the *order* of the determinant, and the products just described are called its *terms*. The determinant may also be represented by the more compact symbol $|a_1\ b_2|$, or $|a_1\ b_2\ c_3|$, and so on.

The determinants of the second and third orders are

1. $$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

2. $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{matrix} a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \\ - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1 \end{matrix}$$

Observe that in 2. the products with + signs are got by following the direction of the "leading diagonal" $a_1b_2c_3$; those with − signs, that of the other diagonal. This is not true of determinants of higher order.

Example 1. Prove the truth of the following:

1. $$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 5 & 2 \end{vmatrix} = 9$$

2. $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

3. $$\begin{vmatrix} a & b & c \\ 0 & 0 & 0 \\ d & e & f \end{vmatrix} = 0$$

4. $$\begin{vmatrix} ka & a & d \\ kb & b & e \\ kc & c & f \end{vmatrix} = k \begin{vmatrix} a & a & d \\ b & b & e \\ c & c & f \end{vmatrix} = 0$$

Example 2. Show that the determinant of the fourth order $|a_1b_2c_3d_4|$ has $4! = 24$ terms, of which half are + and half −. What are the signs of the following terms?

$$a_4b_2c_3d_1 \qquad a_3b_4c_1d_2 \qquad a_2b_1c_4d_3$$

Example 3. Show that the signs of the terms of a determinant may also be found by arranging the factors in the normal order of the column marks and then counting the inversions of the row marks.

199. Properties of determinants. The following properties of determinants are immediate consequences of the definition in § 198.

1. *If the first, second, $\cdots$ rows of a determinant are written as its first, second, $\cdots$ columns, the value of the determinant is not changed.*

2. *If all the elements of a row or column are* 0, *the determinant has the value* 0.

3. *If two of the rows or two of the columns are interchanged, the value of the determinant merely changes sign.*

Thus,
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \ (1) = - \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \ (2)$$

For suppose the factors in the terms of each determinant to be arranged in the order of the rows of that determinant. Then each term of (1) appears in (2) with its first and third factors interchanged, therefore with the number of inversions of its subscripts changed by an odd number, § 197, therefore with its sign changed. Thus for $a_2b_3c_1$ in (1), we have $-c_1b_3a_2$ in (2).

4. *If two of the rows, or two of the columns, be identically the same, the determinant has the value zero.*

For if the value were not 0, it would change sign when the identical rows or columns are interchanged; but the determinant remains unchanged.

5. *If all the elements of a row or a column have a common factor, that factor may be separated and written before the determinant.*

Thus,
$$| ka_1 \ b_2 \ c_3 | = k \, | a_1 \ b_2 \ c_3 |$$

6. *If the corresponding elements of two rows or two columns are proportional, the determinant has the value zero.*

7. *Since* $(a_1 + a'_1)b_2c_3 = a_1b_2c_3 + a'_1b_2c_3$, *and so on, we have*

$$\begin{vmatrix} a_1 + a'_1 & a_2 & a_3 \\ b_1 + b'_1 & b_2 & b_3 \\ c_1 + c'_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a'_1 & a_2 & a_3 \\ b'_1 & b_2 & b_3 \\ c'_1 & c_2 & c_3 \end{vmatrix}$$

Any of the numbers a'_1, b'_1, c'_1 may be 0.

8. *The value of a determinant will not be changed if to the elements of any column* (*or row*) *there be added the corresponding*

elements of any other column (or row) all multiplied by the same number k.

This follows from the theorems 6. and 7. Thus

$$\begin{vmatrix} a_1 + ka_3 & a_2 & a_3 \\ b_1 + kb_3 & b_2 & b_3 \\ c_1 + kc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & a_2 & a_3 \\ kb_3 & b_2 & b_3 \\ kc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example. Prove that the following determinants have the values indicated, shortening the reckoning as much as possible by aid of 6., 7., 8.

1. $\begin{vmatrix} 10 & 8 & 2 \\ 15 & 12 & 3 \\ 20 & 32 & 13 \end{vmatrix} = 0$ 2. $\begin{vmatrix} 2 & 3 & 4 \\ 25 & 38 & 50 \\ 15 & 20 & 27 \end{vmatrix} = -3$ 3. $\begin{vmatrix} 9 & 27 & 18 \\ 15 & 42 & 12 \\ 18 & 51 & 14 \end{vmatrix} = 108$

200. Minors. In any determinant, Δ, cancel the row and column in which some particular element, call it e, lies and then form the determinant of the remaining elements without disturbing their relative positions. The determinant thus got is called the *complementary minor* of e and may be represented by Δ_e.

1. *In the expansion of Δ, the sum of all the terms that involve the leading element a_1 is $a_1\Delta_{a_1}$.*

Thus, in $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$, this sum is $a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix}$

For, apart from sign, each term of Δ that involves a_1 is got by multiplying a_1 by elements taken from the remaining rows and columns of Δ, one and but one from each; in other words, by multiplying a_1 by a term of Δ_{a_1}. Moreover the sign of the Δ term is the same as that of the Δ_{a_1} term since it depends solely on the inversions of the subscripts 2, 3, 4. Conversely, the product of a_1 by any term of Δ_{a_1} is a term of Δ.

2. *If e denote the element in the ith row and kth column of Δ, the sum of all the terms of Δ that involve e is $(-1)^{i+k} e\Delta_e$.*

For we can bring e into the position of leading element, without disturbing the relative positions of the elements outside of the row and column in which e stands, by first interchanging e's row with each preceding row in turn, and then interchanging its column with each preceding column. In carrying out these successive interchanges,

we merely change the sign of the determinant $(i-1)+(k-1)$ times. Hence, if Δ' denote the determinant in its final form, we have

$$\Delta' = (-1)^{i+k-2}\Delta = (-1)^{i+k}\Delta$$

The minor of e in Δ' is the same as in Δ. Hence in Δ' the sum of all terms that involve e is $e\Delta_e$, by 1., and therefore in Δ this sum is $(-1)^{i+k}e\Delta_e$.

We call $(-1)^{i+k}\Delta_e$ the *cofactor* of e in the determinant Δ.

3. *A determinant may be expressed as the sum of the products of the elements of one of its rows (or columns) and their complementary minors, with signs which are alternately plus and minus, or minus and plus.*

Thus, $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

and if Δ be the determinant $|\, a_1 \ b_2 \ c_3 \ d_4 \,|$, we have

$$\Delta = a_1\Delta_{a_1} - a_2\Delta_{a_2} + a_3\Delta_{a_3} - a_4\Delta_{a_4} \quad (1)$$
$$= a_1\Delta_{a_1} - b_1\Delta_{b_1} + c_1\Delta_{c_1} - d_1\Delta_{d_1} \quad (2)$$

with similar developments with respect to the other rows and columns.

4. Hence the value of a determinant of the fourth order may always be found by computing the values of four determinants of the third order. But a simpler method is illustrated in the following example.

$$\begin{vmatrix} 1 & 1 & -1 & 2 \\ 2 & 3 & 1 & 1 \\ -1 & 2 & 2 & 0 \\ 3 & 1 & 1 & 2 \end{vmatrix}_{(1)} = \begin{vmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 3 & -3 \\ 0 & 3 & 1 & 2 \\ 0 & -2 & 4 & -4 \end{vmatrix}_{(2)} = \begin{vmatrix} 1 & 3 & -3 \\ 3 & 1 & 2 \\ -2 & 4 & -4 \end{vmatrix}_{(3)} = -30$$

We get (2) by subtracting from the second, third, and fourth rows of (1) the first row multiplied by 2, -1, and 3; and (2) equals (3) by the theorem in 3. This method is directly applicable to any determinant which has an element 1. And a determinant which has no such element can be transformed into another which has one by aid of the theorems of § 199.

201. Cofactors. 1. The cofactors of $a_1, a_2, \cdots$ in the determinant $\Delta = |\, a_1 \ b_2 \ c_3 \ d_4 \,|$, namely Δ_{a_2}, $-\Delta_{a_1}, \cdots$, are represented by $A_1, A_2, \cdots$. Hence, by § 200, (1), (2),

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 \quad (1)$$
$$= a_1A_1 + b_1B_1 + c_1C_1 + d_1D_1 \quad (2)$$

The sum of the products of the elements of any row or column of a determinant Δ *and their cofactors is equal to* Δ.

2. Since A_1, B_1, C_1, D_1 do not involve a_1, b_1, c_1, d_1, the effect of replacing a_1, b_1, c_1, d_1 by a_2, b_2, c_2, d_2 in (2) is the same as that of replacing the first column of Δ by a_2, b_2, c_2, d_2, thus making two columns equal; hence, § 199, 4.,

$$a_2A_1 + b_2B_1 + c_2C_1 + d_2D_1 = 0 \tag{3}$$

The sum of the products of the elements of a column of a determinant Δ *and the cofactors of the corresponding elements of another column is* 0. *Similarly for the rows.*

202. Linear equations. 1. It is required to solve the following set of simultaneous equations for x, y, z.

$$\begin{aligned} a_1x + a_2y + a_3z &= k \\ b_1x + b_2y + b_3z &= l \\ c_1x + c_2y + c_3z &= m \end{aligned} \tag{1}$$

Let Δ denote the determinant of the coefficients of x, y, z, and let A_1, B_1, $\cdots$ denote the cofactors of a_1, b_1, $\cdots$ in Δ. To eliminate y and z, multiply the first equation by A_1, the second by B_1, the third by C_1, and then add. In the result, the coefficient of x is $a_1A_1 + b_1B_1 + c_1C_1$ or Δ; the coefficients of y and z are 0, § 201, 2.; and the second member is $kA_1 + lB_1 + mC_1$, which represents the determinant $|\, k \;\; b_2 \;\; c_3 \,|$ got by replacing the first column of Δ by k, l, m. We therefore have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} x = \begin{vmatrix} k & a_2 & a_3 \\ l & b_2 & b_3 \\ m & c_2 & c_3 \end{vmatrix} \tag{2}$$

In like manner, we find

$$\Delta \cdot y = |\, a_1 \;\; l \;\; c_3 \,| \quad (3) \qquad\qquad \Delta \cdot z = |\, a_1 \;\; b_2 \;\; m \,| \quad (4)$$

Therefore, if $\Delta \neq 0$, we have, on dividing (2), (3), (4) by Δ,

$$x = \frac{|\, k \;\; b_2 \;\; c_3 \,|}{\Delta} \qquad y = \frac{|\, a_1 \;\; l \;\; c_3 \,|}{\Delta} \qquad z = \frac{|\, a_1 \;\; b_2 \;\; m \,|}{\Delta} \tag{5}$$

We have thus proved that if the equations (1) have any solution, it is (5). That (5) actually is a solution may be shown by substituting (5) in (1).

Thus, if we substitute in $a_1x + a_2y + a_3z - k = 0$ and clear of fractions, we find that the first member becomes the expansion of a determinant of the fourth order whose first two rows are a_1, a_2, a_3, k, and which is therefore 0.

This argument may be extended to a set of n linear equations in n unknown letters and establishes the theorem:

A set of n linear equations in n unknown letters has one and but one solution if the determinant of the coefficients of the unknown letters is not 0.

EXAMPLE 1. Solve the following set of equations by this method:

$$2x - y - 3z = 3 \qquad 4x + 5y + 7z = 7 \qquad 3x + 2y - 8z = 15$$

2. When $\Delta = 0$, it follows from (2), (3), (4) that the set of equations (1) has no solution unless $|\,k\ b_2\ c_3\,|$, $|\,a_1\ l\ c_3\,|$, $|\,a_1\,b_2\,m\,|$ are also 0; but if these three determinants are 0, the set (1) may have infinitely many solutions.

Consider the *matrices*, or rectangular arrays of numbers:

$$A \begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{Vmatrix} \qquad\qquad B \begin{Vmatrix} a_1 & a_2 & a_3 & k \\ b_1 & b_2 & b_3 & l \\ c_1 & c_2 & c_3 & m \end{Vmatrix}$$

From any matrix we may obtain a number of square arrays by crossing out columns, rows, or both. The determinants of all these arrays (including that of the matrix itself when square) are called the *determinants of the matrix.*

The order of the determinant (or group of determinants) of highest order that does not vanish is called the *rank* of the matrix.

Since every determinant of A is a determinant of B, the rank of A cannot exceed that of B. It can be proved that

If the ranks of A and B are the same, the set of equations (1) *has one solution, or infinitely many solutions.*

If the rank of A is less than that of B, the set of equations (1) *has no solution.*

These theorems may be extended[1] to any set of m linear equations in n unknown letters, not only when $m = n$ but also when $m \lessgtr n$.

[1] See Bôcher: Introduction to Higher Algebra, § 16.

Example 2. Show that the set of equations $x + z = k$, $y + z = l$, $2x + y + 3z = m$ has infinitely many solutions, or no solution, according as $2k + l$ is, or is not, equal to m.

3. When k, l, and m are 0, the equations (1) become

$$\begin{aligned} a_1x + a_2y + a_3z &= 0 \\ b_1x + b_2y + b_3z &= 0 \\ c_1x + c_2y + c_3z &= 0 \end{aligned} \tag{6}$$

and are said to be *homogeneous*. Equations (2), (3), (4) become

$$\Delta \cdot x = 0 \quad \Delta \cdot y = 0 \quad \Delta \cdot z = 0 \tag{7}$$

and (7) implies that either $\Delta = 0$ or x, y, and $z = 0$.

In this case the ranks of the matrices A and B are the same. If $\Delta \neq 0$, the rank is 3, and (6) has the single solution (0, 0, 0). If $\Delta = 0$, the rank is 2 or 1, and (6) has infinitely many other solutions. For suppose that the rank is 2 and that $|\, a_1 \; b_2 \,|$ is one of the non-vanishing determinants of the second order. We can then solve the first two equations for x, y in terms of z. The result may be written

$$x : y : z = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} : - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} : \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \tag{8}$$

or

$$x = rC_1 \qquad y = rC_2 \qquad z = rC_3 \tag{9}$$

where r is an arbitrary constant. And these values (9) satisfy the third equation of (6); for, substituting them in its first member, we obtain $r(c_1C_1 + c_2C_2 + c_3C_3) = r\Delta = 0$.

If the rank is 1, every solution of one of the equations will satisfy the other two.

Example 3. Show that the set of equations

$$2x + 3y - 5z = 0, \quad x + y + 6z = 0, \quad 4x + 5y + 7z = 0$$

has the solutions

$$x : y : z = 23 : -17 : -1, \text{ or } x = 23r,\ y = -17r,\ z = -r$$

203. Multiplication of determinants. It can be shown that the product of two determinants Δ and Δ' of the same order n may be expressed in the form of a third determinant Δ'' of order n, obtained as follows:

Multiply the elements of the ith row of Δ by the corresponding elements of the kth column of Δ′. The sum of the products thus obtained is the element in the ith row and kth column of Δ″.

Thus, $$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = \begin{vmatrix} a_1p_1 + a_2q_1 & a_1p_2 + a_2q_2 \\ b_1p_1 + b_2q_1 & b_1p_2 + b_2q_2 \end{vmatrix}$$

For, by § 199, 7., the third determinant is the sum of

$$\begin{vmatrix} a_1p_1 & a_1p_2 \\ b_1p_1 & b_1p_2 \end{vmatrix} (1) \qquad \begin{vmatrix} a_1p_1 & a_2q_2 \\ b_1p_1 & b_2q_2 \end{vmatrix} (2) \qquad \begin{vmatrix} a_2q_1 & a_1p_2 \\ b_2q_1 & b_1p_2 \end{vmatrix} (3) \qquad \begin{vmatrix} a_2q_1 & a_2q_2 \\ b_2q_1 & b_2q_2 \end{vmatrix} (4)$$

But (1) and (4) are 0, by § 199, 6., and the sum of (2) and (3) is

$$p_1q_2 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + p_2q_1 \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}$$

EXERCISE XL

1. If the elements of $| a_1 \; b_2 \; c_3 |$ are functions of x, show that

$$\frac{d}{dx}\begin{vmatrix} a_1 & b_2 & c_3 \end{vmatrix} = \begin{vmatrix} \dfrac{da_1}{dx} & b_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \dfrac{db_2}{dx} & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_2 & \dfrac{dc_3}{dx} \end{vmatrix}$$

2. Express $| a_1b_2c_3 | \cdot | p_1q_2r_3 |$ as a determinant of the third order.

3. Show, by eliminating x, y as in § 202, that the set of equations

$$a_1x + a_2y + a_3 = 0 \qquad b_1x + b_2y + b_3 = 0 \qquad c_1x + c_2y + c_3 = 0$$

has no solution unless $\Delta = | a_1b_2c_3 | = 0$. If $\Delta = 0$, when does the set have one solution; no solution; infinitely many solutions?

4. If $ax + by + cz = 0$, $a'x + b'y + c'z = 0$, and $a : b : c \neq a' : b' : c'$, show that

$$x : y : z = \begin{vmatrix} b & c \\ b' & c' \end{vmatrix} : \begin{vmatrix} c & a \\ c' & a' \end{vmatrix} : \begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$$

5. The coefficients A, B, C of the equation of the straight line determined by two points (x_1, y_1), (x_2, y_2) satisfy the equations

$$Ax + By + C = 0 \qquad Ax_1 + By_1 + C = 0 \qquad Ax_2 + By_2 + C = 0$$

Show that solving the second and third of these equations for $A : B : C$ and substituting the result in the first gives

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \qquad (1)$$

Show also by § 199, 4. that (1) is the equation of the line.

6. By the reasoning in § 202, 3., show that the set of equations § 202, (1) has infinitely many solutions when the matrices A and B are both of rank 2, but no solution when A is of rank 2, and B of rank 3.

XXII. SOLID GEOMETRY

204. Space coordinates. Through the point O chosen as origin, take three mutually perpendicular lines as *axes of coordinates in space*, Ox and Oz in the plane of the paper, and Oy perpendicular to that plane; also take as the positive directions along Ox, Oy, Oz those indicated by the arrow heads in the figure.

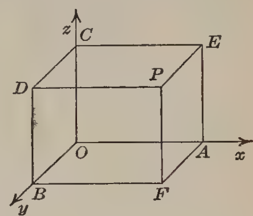

FIG. 110.

The axes Ox, Oy, Oz determine three mutually perpendicular planes, the xy-, yz-, and xz-planes. They are called the *coordinate planes.*

The distances of any point P from the yz-, zx-, and xy-planes are denoted by x, y, and z respectively, and are called the *coordinates* of P.

By taking planes through P parallel to the coordinate planes, construct the parallelepiped in Fig. 110.

The points D, E, F are the projections of P on the yz-, zx-, xy-planes. The points A, B, C are the projections of P on the x-, y-, z-axes.

$$x = DP = OA \qquad y = EP = OB \qquad z = FP = OC$$

In Fig. 110 the directions of DP, EP, FP are positive, and the signs of x, y, z therefore plus. But when P is to the left of the yz-plane, or behind the zx-plane, or below the xy-plane, the sign of x, or y, or z is minus.

Conversely, let a, b, c denote any given set of values of x, y, z. The point $P(a, b, c)$ of which a, b, c are the x-, y-, z-coordinates can be got by taking $OA = a$, then $AF = b$, then $FP = c$.

205. Equations in x, y, z and their loci. For points in the xy-plane, and such points only, we have $z = 0$. Hence $z = 0$ is called the equation of the xy-plane, and the xy-plane is called the locus of the equation $z = 0$. Similarly $z = 2$ is the equation of the plane through $(0, 0, 2)$ parallel to the

xy-plane. The like is true of equations of the forms $x = a$ and $y = b$.

For points on the line Oz, and such points only, we have $x = 0$, $y = 0$; hence $x = 0$, $y = 0$ are called the equations of Oz, and Oz is called the locus of $x = 0$, $y = 0$. Similarly for $x = a$, $y = b$; and so on.

An equation in x, y, z which is true for every point of a given set of points, and for such points only, is called the equation of the set; and the set is called the locus of the equation.

Similarly for a pair of simultaneous equations in x, y, z.

Generally speaking, the locus of an algebraic equation, $f = 0$, in x, y, z is a surface — a plane when the equation is of the first degree, § 213; and the locus of a pair of simultaneous equations, $f = 0$, $\phi = 0$, is the curve or curves, if any, in which the surfaces $f = 0$, $\phi = 0$ intersect.

The shape of a surface $f = 0$ may be learned from its sections by planes parallel to the coordinate planes, as is illustrated in Ex. 1 on p. 158.

EXAMPLE 1. Let the curve C in Fig. 111 be the locus of $f(x, y) = 0$ *in the xy-plane.* If through any point Q of C a line l be taken parallel to Oz, the x, y coordinates of every point P of l will be the same as those of Q and will therefore satisfy $f(x, y) = 0$. Hence the locus of $f(x, y) = 0$ *in space* is the *cylindrical surface* through C whose elements are parallel to Oz. In space, the curve C is represented by the pair of equations $f(x, y) = 0$, $z = 0$.

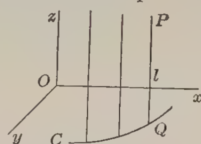

FIG. 111.

Similarly for equations of the forms $f(y, z) = 0$ and $f(z, x) = 0$.

EXAMPLE 2. What are the loci in space of the following equations?

1. $x - y = 0$ **2.** $y^2 = x$ **3.** $y + 3z = 4$ **4.** $x^2 + z^2 = a^2$

EXAMPLE 3. The locus of $x^2/a^2 - y^2/b^2 = z$ is a saddle shaped surface called the *hyperbolic paraboloid,* Fig. 112. Its sections by planes $x = k$ are parabolas $z = - y^2/b + k^2/a^2$, $x = k$ which open downwards. Its sections by planes $y = k$ are parabolas which open upwards. Its

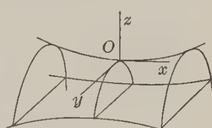

FIG. 112.

section by the plane $z = 0$ is the pair of lines $x/a + y/b = 0$, $z = 0$ (1) and $x/a - y/b = 0$, $z = 0$ (2). Its sections by planes $z = k (k \neq 0)$ are hyperbolas with asymptotes parallel to the lines (1) and (2).

EXAMPLE 4. The locus of the equation $x^2/a^2 + y^2/b^2 = z$ is an *elliptic paraboloid.* Sketch this surface.

206. Length and direction cosines of *OP*. 1. Let P be the point (x, y, z).

We have $OP^2 = OF^2 + FP^2$, $OF^2 = OA^2 + AF^2$

Hence $\quad OP^2 = x^2 + y^2 + z^2 \qquad (1)$

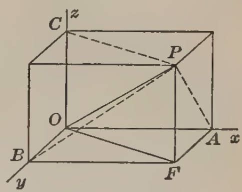

FIG. 113.

2. The cosines of the angles POx, POy, POz, which the direction OP, from O to P, makes with the positive directions on the x-, y-, z-axes, are called the *direction cosines* of OP. Let α, β, γ denote the angles, and l, m, n their cosines.

Since OAP, OBP, OCP are right angles, we have

$$\cos\alpha = x/OP \qquad \cos\beta = y/OP \qquad \cos\gamma = z/OP \quad (2)$$

Hence, by (1),

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (x^2 + y^2 + z^2)/OP^2 = 1$$

The direction cosines l, m, n, of any direction OP are connected by the relation

$$l^2 + m^2 + n^2 = 1 \qquad (3)$$

The direction angles of the direction PO are $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$; hence its direction cosines are $-l$, $-m$, $-n$.

EXAMPLE 1. Find l, m, n for the direction from O to P $(2, 3, -6)$.

$$OP = (4 + 9 + 36)^{1/2} = 7 \quad \therefore l = 2/7, m = 3/7, n = -6/7$$

EXAMPLE 2. Find l, m, n for the positive and negative directions on Ox, Oy, and Oz.

207. Direction cosines of a line. Let L denote any given line. On the parallel to L through O, take OP (Fig. 113) in either of the directions along L. The l, m, n of this OP (or those of PO) are called the *direction cosines* of L.

The direction cosines l, m, n of L are determined, apart from sign, when their ratios are given; hence these ratios may be called the *direction ratios* of L.

For if $a : b : c$ are the given ratios, then $l = ka$, $m = kb$, $n = kc$, where k is some constant and, by § 206 (3),

$$k^2(a^2 + b^2 + c^2) = l^2 + m^2 + n^2 = 1$$

Hence $k = 1/\pm(a^2 + b^2 + c^2)^{1/2}$, and therefore, for one of the directions on L,

$$l, m, n = a, b, c, \text{ each divided by } (a^2 + b^2 + c^2)^{1/2} \qquad (1)$$

Thus, if $l : m : n = 1 : 2 : 2$, then $l = 1/3$, $m = 2/3$, $n = 2/3$.

208. Projections of directed line segments. Let L be a given line, and AB a directed segment of another line L'. Let planes taken through A and B perpendicular to L cut L at A_0 and B_0. The segment A_0B_0 is called the *projection of* AB *on* L.

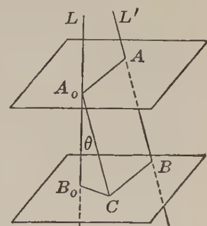

FIG. 114.

If positive directions be chosen on L and L', and these directions make the angle θ, then

$$A_0B_0 = AB \cdot \cos\theta \qquad (1)$$
$$B_0A_0 = BA \cdot \cos\theta$$

209. Length and direction cosines of $P'P''$. Let $P'(x', y', z')$ and $P''(x'', y'', z'')$ be two given points; the length s and the direction cosines l, m, n of the directed line segment $P'P''$ may be found as follows:

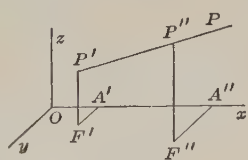

FIG. 115.

The projection of $P'P''$ on Ox is $A'A'' = x'' - x'$. Similarly, the projections of $P'P''$ on Oy and Oz are $y'' - y'$ and $z'' - z'$. Hence, by § 208 (1),

$$P'P'' \cdot l = x'' - x' \qquad P'P'' \cdot m = y'' - y' \qquad P'P'' \cdot n = z'' - z' \qquad (1)$$

Squaring, adding, and applying § 206 (3), we get

$$P'P''^2 = (x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2$$

Hence

$$s = [(x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2]^{1/2} \qquad (2)$$

$$l = (x'' - x')/s \qquad m = (y'' - y')/s \qquad n = (z'' - z')/s \qquad (3)$$

210. Equations of a line. By § 209 (1), if L denote a given line, P' and P'' any two of its points, and l, m, n its direction cosines, then

$$(x'' - x') : (y'' - y') : (z'' - z') = l : m : n \tag{1}$$

Hence the equations of a line L through a given point P' and having given direction ratios $a : b : c$ are

$$\frac{x - x'}{a} = \frac{y - y'}{b} = \frac{z - z'}{c} \tag{2}$$

For (2) is true for every point $P(x, y, z)$ of L and for such points only.

EXAMPLE 1. Show that the equations of the line through P' and P'' are

$$\frac{x - x'}{x' - x''} = \frac{y - y'}{y' - y''} = \frac{z - z'}{z' - z''} \tag{3}$$

and find these equations when P' and P'' are (2, 3, 5) and (4, $-$ 1, 6).

EXAMPLE 2. Representing $P'P$ by r, show by § 209 (1) that the line through P' and having the direction cosines l, m, n, has the parametric equations

$$x = x' + lr \qquad y = y' + mr \qquad z = z' + nr \tag{4}$$

211. Angle between two given directions. Let the direction cosines of OP and OP' be l, m, n and l', m', n'; and let θ be the angle between OP and OP'. Make the construction in Fig. 116. Since O is joined to P by OP, and also by the broken line $OAFP$, the projection OD of OP on OP' equals the algebraic sum of the projections of OA, AF, FP on OP'. Hence, § 208 (1),

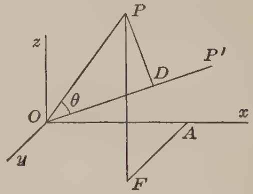

FIG. 116.

$$\begin{aligned} OP \cos\theta &= OA \cdot l' + AF \cdot m' + FP \cdot n' \\ &= OP \cdot ll' + OP \cdot mm' + OP \cdot nn' \end{aligned}$$

Therefore
$$\cos\theta = ll' + mm' + nn' \tag{1}$$

212. Condition of perpendicularity. Since $\cos \pi/2 = 0$, it follows from § 211 (1) and § 207 (1) that the condition

that two lines whose direction ratios are $a:b:c$ and $a':b':c'$ shall be perpendicular is that

$$aa' + bb' + cc' = 0 \tag{1}$$

Thus the lines joining O to P $(2, 3, -2)$ and P' $(3, 4, 9)$ are perpendicular; for $2 \cdot 3 + 3 \cdot 4 - 2 \cdot 9 = 0$.

213. Equation of a plane. 1. Let $P'(x', y', z')$ be a known point of a given plane E, and let $a:b:c$ be the direction ratios of the normals to E.

Take any representative point $P(x, y, z)$ of E. By § 210 (1), the direction ratios of the line PP' are $(x - x'):(y - y'):(z - z')$. Therefore, since the normals to E are perpendicular to PP', we have, § 212 (1),

$$a(x - x') + b(y - y') + c(z - z') = 0 \tag{1}$$

Since (1) is true for every point $P(x, y, z)$ of the plane, and for such points only, it is the equation of the plane.

2. Every equation of the first degree in x, y, z,

$$Ax + By + Cz + D = 0 \tag{2}$$

represents a plane. For let x', y', z' be any solution of (2) so that

$$Ax' + By' + Cz' + D = 0 \tag{3}$$

Subtracting (3) from (2), we get

$$A(x - x') + B(y - y') + C(z - z') = 0$$

which is of the form (1). Hence the equation (2) represents a plane, and $A:B:C$ are the direction ratios of the normals to this plane.

Example 1. Show that the plane through the point $(1, 0, -2)$ and parallel to the plane $3x - 8y + 2z = 0$ has the equation $3(x - 1) - 8y + 2(z + 2) = 0$.

Example 2. Plot the plane $12x + 10y + 15z - 60 = 0$.

Setting $y = 0$, $z = 0$, we get $x = 5$; hence the x-intercept is 5. Similarly, the y-intercept is 6, and the z-intercept 4. Hence the plane is that marked ABC in Fig. 117.

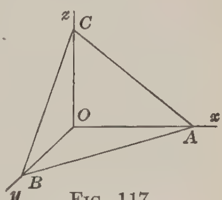

Fig. 117.

EXAMPLE 3. Plot the plane $2x - 2y - z = 0$ by aid of its traces on the xy- and xz- planes.

EXAMPLE 4. Find the direction ratios of the line of intersection of the planes $2x - y + z = 0$, $x + y + z = 0$.

The line is perpendicular to the normals to both planes. Hence, its l, m, n satisfy the equations $2l - m + n = 0$, $l + m + n = 0$, § 212 (1), and therefore, p. 229, Ex. 4,

$$l : m : n = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} : \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} : \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -2 : -1 : 3$$

EXERCISE XLI

1. Show that the equation of a sphere with center (a, b, c) and radius r is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

2. Find the center and radius of the sphere

$$x^2 + y^2 + z^2 - 4x + 5z = 0.$$

3. Find the length and direction cosines of the vector from the origin to the point $(3, -4, 12)$; also the lengths of its projections on the yz-, zx-, and zy-planes.

4. Find the cosine of the angle between two lines whose direction ratios are $1 : 2 : 2$ and $3 : 0 : 4$.

5. Find the l, m, n of a line perpendicular to two lines whose direction ratios are $2 : -1 : 1$ and $1 : 2 : 3$. If the line passes through $(0, 1, -1)$, what are its equations.

6. Find the length of the projection of the vector from $(0, 4, 1)$ to $(1, 2, -1)$ upon a line whose direction ratios are $2 : 3 : -6$; upon the plane $7x - 2y + 4z = 5$.

7. Find the equation of the plane through $(1, 0, -2)$ and perpendicular to the line of intersection of the planes $3x + 4y + 7z = 0$, and $x - y + 2z = 0$.

8. What determinant equated to 0 represents the plane determined by three given points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)?

9. Eliminating z between $x^2 + y^2 + z^2 = 4$ (1) and $z = y$ (2) gives $x^2 + 2y^2 = 4$ (3). Show that (3) and $z = 0$ are the equations of the projection on the xy-plane of the circle in which the plane (2) cuts the sphere (1).

10. Show that the equation of the conical surface got by revolving the line $y = 2x$, $z = 0$ about Ox may be found by replacing y by $\sqrt{y^2 + z^2}$ in $y = 2x$ and then rationalizing, which gives

$$y^2 + z^2 - 4x^2 = 0.$$

11. Show that if x_1, y_1, z_1 is any real solution of a homogeneous equation in x, y, z, as $x^2 - xy + 3z^2 = 0$ (1), so also is kx_1, ky_1, kz_1; hence that the line through O and the point (x_1, y_1, z_1) lies wholly in the surface (1), which is therefore a conical surface.

12. Through O' (a, b, c), and parallel to Ox, Oy, Oz (1), take $O'x'$, $O'y'$, $O'z'$ (2). Show that if the coordinates of any point P referred to the axes (1) and (2) are x, y, z and x', y', z', then

$$x = x' + a \qquad y = y' + b \qquad z = z' + c \tag{1}$$

13. Let Ox, Oy, Oz (1) and Ox', Oy', Oz' (2) be two sets of rectangular axes through O, and let (x, x') denote the angle between Ox and Ox', and so on. By the method of projection used in § 211, show that if the coordinates of any point P referred to (1) and (2) are x, y, z and x', y', z', then

$$\begin{aligned} x' &= x\cos(x, x') + y\cos(y, x') + z\cos(z, x') \\ y' &= x\cos(x, y') + y\cos(y, y') + z\cos(z, y') \\ z' &= x\cos(x, z') + y\cos(y, z') + z\cos(z, z') \end{aligned} \tag{2}$$

What are the corresponding expressions for x, y, z in terms of x', y', z'?

14. By aid of the formulas in Exs. 12, 13, justify the footnote respecting centroids on p. 178.

15. By § 203, prove that the square of the determinant of the coefficients of x, y, z in the equations (2) of Ex. 13 equals 1.

XXIII. PARTIAL DIFFERENTIATION

214. Continuous functions of two or more variables. 1. In the case of any number of independent variables $x, y, \cdots$, as when the number is one, two, or three, we call any particular set of values of the variables, as $(a, b, \cdots)$, a *point*.

2. The set of all points $(x, y, \cdots)$ got by giving x any value in a certain number interval (α_1, β_1), y any value in a second interval (α_2, β_2), and so on, may be called a *region* R.

3. Let $f(x, y, \cdots)$ denote a *one-valued function* of $x, y, \cdots$ in R, so that to each point of R corresponds a single real value of $f(x, y, \cdots)$.

4. We say that $f(x, y, \cdots)$ is *continuous*[1] *at the point* $(a, b, \cdots)$ of R, if $f(a, b, \cdots)$ is finite, and if

$$f(x, y, \cdots) \rightarrow f(a, b, \cdots) \text{ when } (x, y, \cdots) \rightarrow (a, b, \cdots) \text{ in } R$$

5. We say that $f(x, y, \cdots)$ is *continuous in* R when it is continuous at all points of R, its boundary included.

6. We say that $f(x, y, \cdots)$ is *continuous in the neighborhood*[2] *of the point* $(a, b, \cdots)$ if it is continuous in a region S consisting of all points $(x, y, \cdots)$ for which $|x - a|, |y - b|, \cdots$ are each less than some positive number d.

[1] Obviously a continuous function of two or more independent variables is a continuous function of the variables taken separately. But the converse is not always true. Thus, the function $f(x, y)$, defined as equal to $xy/(x^2+y^2)$ when x, y are not both 0, and as 0 when x, y are both 0, is a continuous function of either variable when the other has any constant value, 0 included. But $f(x, y)$ is not a continuous function of x, y at 0. If, for example, $(x, y) \rightarrow 0$ on the line $y = 2x$, then $f(x, y) \rightarrow 2/5$, and not 0.

[2] A function of one or more variables may be continuous at a point but not continuous in the neighborhood of the point. Thus the function $f(x)$ which is defined as equal to x when x is irrational, but as equal to 0 when x is rational, is continuous at 0, but not in its neighborhood.

215. Partial derivatives. The result of differentiating $u = f(x, y)$ with respect to x, treating y as a constant, is called the *partial derivative of u with respect to x* and is denoted by one of the symbols $\partial u/\partial x$, $\partial f/\partial x$, or $f_x(x, y)$. Hence, by definition,

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

and $\partial f/\partial x$ exists at any point (x, y) when and only when this limit exists at that point.

The partial derivative of $f(x, y)$ with respect to y is similarly defined; so also are the partial derivatives of $f(x, y, z)$ with respect to x, y, z; and so on.

216. Total increments and differentials. 1. Let $f(x, y)$, f_x, and f_y be continuous at the point (x, y) and in its neighborhood, that is, in a region S about (x, y) of the kind described in § 214, 6. Give x, y any increments $\Delta x, \Delta y$ such that the point $(x + \Delta x, y + \Delta y)$ also is in S. Then $u = f(x, y)$ receives the increment

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y) \tag{1}$$

Subtract and add $f(x, y + \Delta y)$; we obtain

$$\begin{aligned}\Delta u = {} & [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] \\ & + [f(y, y + \Delta y) - f(x, y)]\end{aligned} \tag{2}$$

The first bracketed expression is the increment which $f(x, y + \Delta y)$ receives when x is given the increment Δx. Hence, by the mean value theorem, § 97 (6),

$$f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) = f_x(x+\theta_1 \Delta x, y+\Delta y)\Delta x \tag{3}$$

where θ_1 is some number between 0 and 1. Similarly

$$f(x, y+\Delta y) - f(x, y) = f_y(x, y+\theta_2 \Delta y)\Delta y \qquad 0<\theta_2<1 \tag{4}$$

Substituting (3) and (4) in (2), we obtain

$$\Delta u = f_x(x + \theta_1 \Delta x, y + \Delta y)\Delta x + f_y(x, y + \theta_2 \Delta y)\Delta y \tag{5}$$

Since f_x and f_y are continuous in S, the coefficients of Δx and Δy in (5) $\to f_x(x, y)$ and $f_y(x, y)$ when $\Delta x, \Delta y \to 0$, and

therefore differ from $f_x(x, y)$ and $f_y(x, y)$ by infinitesimals ϵ_1, ϵ_2, which $\to 0$ when $\Delta x, \Delta y \to 0$. Hence

$$\Delta u = \left[\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right] + [\epsilon_1 \Delta x + \epsilon_2 \Delta y] \tag{6}$$

Ordinarily $[\epsilon_1 \Delta x + \epsilon_2 \Delta y]/[f_x \Delta x + f_y \Delta y] \to 0$ when $\Delta x, \Delta y \to 0$, that is, $f_x \Delta x + f_y \Delta y$ is the *principal part* of Δu, and represents Δu approximately for small values of $\Delta x, \Delta y$. It is called the *total differential* of u and is denoted by du; so that, by definition,

$$du = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y \tag{7}$$

When $u = x$, (7) becomes $dx = \Delta x$; and when $u = y$, it becomes $dy = \Delta y$. Hence (7) implies that the total differentials of the independent variables x, y are the same as their increments, and it may also be written

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{8}$$

2. We can deal in a similar manner with the increment of a function of more than two variables. Thus if the function $u = f(x, y, z)$, and f_x, f_y, f_z are continuous at and in the neighborhood of the point (x, y, z), we can reduce Δu to a form, like (6), whose principal part, the differential of u, is given by the formula

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \tag{9}$$

Example 1. If $u = x^2y^3z$, then $du = 2\,xy^3z\,dx + 3\,x^2y^2z\,dy + x^2y^3\,dz$.

Example 2. A rectangle whose sides have the lengths x, y has the area $u = xy$. Hence, by (8), the change in the area due to small changes dx, dy, in the lengths of the sides is approximately $du = y\,dx + x\,dy$. Or, if x, y denote the measurements made of the sides, and dx, dy the errors in the measuring, du is the approximate error in the area. Thus, if the measurements are $x, y = 5, 6$ with possible errors $dx = dy = \pm .01$, the approximate maximum error is $du = 6(.01) + 5(.01) = .11$.

Example 3. Find the total differentials of y/x, xy/z, and $\sin xyz$.

EXAMPLE 4. What approximately is the change in $(x^2 + y^2)^{1/2}$ when x, y change from 3, 4, to 3.01, 3.99?

EXAMPLE 5. A steamship is supposed to have made 40 miles in 2 hours. If the estimates of the distance and time may have been in error by 100 ft. and 2 min., how much may the actual average speed have differed from 20 m./h.?

EXAMPLE 6. If small changes be made in the lengths x, y of two sides of a triangle and in the included angle θ, what approximately is the corresponding change in the area?

217. Derivatives of composite functions. Let the variables x, y in $u = f(x, y)$ be differentiable functions of another variable t, and let the increments Δx, Δy in § 216 be those which correspond to the increment Δt of t. Dividing both members of § 216 (5) by Δt, we have

$$\frac{\Delta u}{\Delta t} = f_x(x + \theta_1 \Delta x, y + \Delta y)\frac{\Delta x}{\Delta t} + f_y(x, y + \theta_2 \Delta y)\frac{\Delta y}{\Delta t}$$

and therefore, making $\Delta t \to 0$,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \tag{1}$$

Similarly, if x, y are functions of t and other variables,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \tag{2}$$

In like manner if $u = f(x, y, z)$, we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \tag{3}$$

and the corresponding formula for $\partial u/\partial t$; and so on.

EXAMPLE. If $u = ax + by$, and $x = a_1x' + b_1y'$, $y = a_2x' + b_2y'$, then $\dfrac{\partial u}{\partial x'} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial x'} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial x'} = aa_1 + ba_2$. Similarly, $\dfrac{\partial u}{\partial y'} = ab_1 + bb_2$.

EXERCISE XLII

1. If $u = x(y - z)$, and $x = r + 2s$, $y = s - t$, $z = t$, find

$$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}.$$

2. If $u = f(x, y, z)$, $x = a_1x' + b_1y' + c_1z'$, $y = a_2x' + b_2y' + c_2z'$, and $z = a_3x' + b_3y' + c_3z'$, find $\partial u/\partial x'$, $\partial u/\partial y'$, $\partial u/\partial z'$.

3. Let u be a function of x, y. If $x = r\cos\theta$ and $y = r\sin\theta$, find $\partial u/\partial r$ and $\partial u/\partial\theta$, and show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

4. If the edges x, y, z of a rectangular parallelopiped are increasing at the rates 2, 3, 4 in./sec., at what rate is the volume increasing when $x, y, z = 5, 6, 7$?

5. Representing the derivative of $f(ax + by)$ with respect to $ax + by$ by $f'(ax + by)$, show that

$$\frac{\partial f}{\partial x} = af', \quad \frac{\partial f}{\partial y} = bf', \text{ and therefore } b\frac{\partial f}{\partial x} = a\frac{\partial f}{\partial y}$$

6. Let u and v be functions of x, y. Show that if $v = f(u)$, then

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0$$

7. Show that if x, y, z are independent of t, then

$$\frac{d}{dt}f(x, ty, t^2z) = yf_{(ty)}(x, ty, t^2z) + 2\,tzf_{(t^2z)}(x, ty, t^2z).$$

8. A function $f(x, y, z)$ is said to be *homogeneous* with respect to x, y, z if it satisfies an identity of the form

$$f(tx, ty, tz) \equiv t^n f(x, y, z) \qquad (1)$$

Thus $f(x, y, z) = (x^3 + y^3)/z$ is homogeneous; $f(tx, ty, tz) = t^2 f(x, y, z)$.

By differentiating (1) with respect to t and setting $t = 1$ in the result, show that

$$nf(x, y, z) \equiv xf_x(x, y, z) + yf_y(x, y, z) + zf_z(x, y, z) \qquad (2)$$

This identity is called *Euler's theorem* respecting homogeneous functions. The definition and theorem apply to functions of any number of variables.

9. Show that the following functions are homogeneous and verify Euler's theorem for them:

1. $x\sin^{-1}(y^2/x^2)$ 2. $y + x(\log x - \log y)$ 3. $xyz/(x + y + z)$

10. If $z = \dfrac{1}{x}f\left(\dfrac{y}{x}\right)$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} + z = 0$

218. Differentials of composite functions. 1. Multiplying § 217 (1) by dt gives

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

which is the formula (8) of § 216. Hence this formula continues to hold good when, instead of being independent

variables, x, y are differentiable functions of another variable. The like is true when they are functions of two or more other variables.

Thus let $u = f(x, y)$, where $x = \phi(s, t)$, $y = \psi(s, t)$, and $f_x, f_y, \phi_s, \phi_t, \psi_s, \psi_t$ are continuous. Since u is a function of s, t, and s, t are independent variables,

$$du = \frac{\partial u}{\partial s} ds + \frac{\partial u}{\partial t} dt \tag{1}$$

$$dx = \frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt \quad (2) \qquad dy = \frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt \quad (3)$$

Multiply (2) by $\partial u/\partial x$ and (3) by $\partial u/\partial y$, and add; we get

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\right) ds + \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial t}\right) dt \tag{4}$$

By § 217 (2), the coefficients of ds and dt in (4) are $\partial u/\partial s$ and $\partial u/\partial t$. Hence, by (1), the right member of (4), and therefore the left, equals du; that is,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \tag{5}$$

It also follows from this proof that in any given case we can find the equation (1) by substituting in (5) the values of dx and dy given by (2) and (3).

EXAMPLE 1. If $u = 3x + 4y$, $x = 2s - t$, $y = 5t$, find du in terms of ds, dt.

$$du = 3\,dx + 4\,dy = 3(2\,ds - dt) + 4(5\,dt) = 6\,ds + 17\,dt.$$

Hence $\partial u/\partial s = 6$, $\partial u/\partial t = 17$. Verify this by using § 217 (2).

2. By similar reasoning, we can prove the general theorem:

If $u = f(x, y, \cdots)$, then $du = f_x\,dx + f_y\,dy + \cdots$, whether $x, y, \cdots$ are independent variables or functions of other variables, provided that all the partial derivatives involved exist and are continuous.

In particular, the formulas of differentiation, as

$$df(u) = f'(u)\,du \qquad d(uv) = v\,du + u\,dv$$

hold good when u, v are independent or are functions of other variables.

EXAMPLE 2. $d \log (x^2+y) = \dfrac{d(x^2+y)}{x^2+y} = \dfrac{2x\,dx+dy}{x^2+y} = \dfrac{2x\,dx}{x^2+y} + \dfrac{dy}{x^2+y}$

3. *When $x, y, \cdots$ are independent variables, and it has been shown in any way respecting a certain function $f(x, y, \cdots)$ that $df = P(x, y, \cdots)\, dx + Q(x, y, \cdots)\, dy + \cdots$, it may be inferred that $P = f_x, Q = f_y, \cdots$.*

For we also have $df = f_x\, dx + f_y\, dy + \cdots$. Hence

$$P\, dx + Q\, dy + \cdots = f_x\, dx + f_y\, dy + \cdots$$

Since $x, y, \cdots$ are independent, $dx, dy, \cdots$ are arbitrary. Hence we may equate all of them except dx to 0, which gives $P\, dx = f_x\, dx$, and therefore $P = f_x$; and so on.

Thus, from Ex. 2 it follows for $u = \log (x^2 + y)$ that

$$\frac{\partial u}{\partial x} = \frac{2\,x}{x^2 + y}, \quad \frac{\partial u}{\partial y} = \frac{1}{x^2 + y}.$$

EXERCISE XLIII

1. From $de^{xy^2} = e^{xy^2} d(xy^2)$, find $\partial e^{xy^2}/\partial x$ and $\partial e^{xy^2}/\partial y$.

2. Find the partial derivatives of $\tan^{-1}(y/x)$ and $\sin (2\,x - y + 3\,z)$.

3. Find the total differentials of each of the following:

   1. $(x^2 + y^2)^{1/2}$ 2. $(x + y)/(x + z)$ 3. $e^x \sin (y + 2\,z)$

4. If $u = x^2 - 3\,y^2$, $x = 4\,s + 5\,t$, $y = t^2$, find du in terms of ds, dt.

5. If $u = xy/z$, $x = 2 \cos \theta$, $y = \sin \theta$, and $z = 3\,\phi$, find du in terms of $d\theta$, $d\phi$.

6. Show that if $x = r \cos \theta$, $y = r \sin \theta$, then $x\, dy - y\, dx = r^2\, d\theta$.

219. Partial derivatives of the second order. The partial derivatives of $\partial f/\partial x$, $\partial f/\partial y$ are called the partial derivatives of the second order of $f(x, y)$.

They are represented as follows:

$\frac{\partial}{\partial x}\frac{\partial f}{\partial x}$, by $\frac{\partial^2 f}{\partial x^2}$ or $f_{xx}(x, y)$ $\qquad$ $\frac{\partial}{\partial y}\frac{\partial f}{\partial x}$, by $\frac{\partial^2 f}{\partial y\, \partial x}$ or $f_{yx}(x, y)$

$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}$, by $\frac{\partial^2 f}{\partial x\, \partial y}$ or $f_{xy}(x, y)$ $\qquad$ $\frac{\partial}{\partial y}\frac{\partial f}{\partial y}$, by $\frac{\partial^2 f}{\partial y^2}$ or $f_{yy}(x, y)$

EXAMPLE. If $u = x^2y^3$, show that

$$\frac{\partial^2 u}{\partial x\, \partial y} = 6\,xy^2, \quad \frac{\partial^2 u}{\partial y\, \partial x} = 6\,xy^2 \quad \therefore \frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}.$$

220. Theorem. *If f_x, f_y, f_{xy}, f_{yx} are continuous at the point (x, y) and in its neighborhood, then*

$$f_{xy}(x, y) = f_{yx}(x, y)$$

By hypothesis, f_x, f_y, f_{xy}, f_{yx} are continuous in a region S about (x, y) of the kind described in § 214, 6. Suppose that h, k are such that the point $(x + h, y + k)$ belongs to S, and consider the expression

$$U = f(x + h, y + k) - f(x + h, y) - f(x, y + k) + f(x, y) \qquad (1)$$

For convenience, set $F(y) = f(x + h, y) - f(x, y)$ (2)

we then have

$$\begin{aligned} U &= F(y + k) - F(y) \\ &= F_y(y + \theta_1 k)k && \text{[by § 97 (6)]} \\ &= [f_y(x + h, y + \theta_1 k) - f_y(x, y + \theta_1 k)]k \\ &= f_{xy}(x + \theta_2 h, y + \theta_1 k)hk && \text{[by § 97 (6)]} \quad (3) \end{aligned}$$

Similarly, by setting $G(x) = f(x, y + k) - f(x, y)$, we find

$$U = f_{yx}(x + \theta_3 h, y + \theta_4 k)kh \qquad (4)$$

By (3) and (4), we have

$$f_{xy}(x + \theta_2 h, y + \theta_1 k) = f_{yx}(x + \theta_3 h, y + \theta_4 k) \qquad (5)$$

and since f_{xy}, f_{yx} are continuous, it follows from (5), when $h, k \to 0$, that

$$f_{xy}(x, y) = f_{yx}(x, y) \qquad (6)$$

221. Partial derivatives of higher order. It follows from § 220 that all the partial derivatives that can be got from $u = f(x, y)$ by differentiating m times with respect to x and n times with respect to y in any order are equal, provided that all the partial derivatives involved in the process are continuous. Hence they may be represented by the same symbol. Using the symbols of operation $\partial/\partial x$, $\partial/\partial y$, we write

$$\left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n u = \frac{\partial^{m+n}}{\partial x^m\, \partial y^n} u = \frac{\partial^{m+n} u}{\partial x^m\, \partial y^n}$$

The like is true of the partial derivatives of $u = f(x, y, z)$.

EXERCISE XLIV

1. Verify the theorem of § 220 for $u = x^2 \sin 2y$ and $u = xy^2z^3$.

2. Show that if $u = \log (x^2 + y^2)^{1/2}$, then $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

3. Show that if $u = (x^2 + y^2 + z^2)^{-1/2}$, then $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

4. If $u = x^2 - y^2$ and $x = 3s + 2t$, $y = -s + 4t$, find $\partial^2 u/\partial s^2$, and $\partial^2 u/\partial t^2$.

5. Show that if $u = f(y + ax) + \phi(y - ax)$, then $\dfrac{\partial^2 u}{\partial x^2} = a^2 \dfrac{\partial^2 u}{\partial y^2}$.

6. If $f(x, y)$ is homogeneous and of degree n, show that

$$n(n-1)f(x, y) = f_{xx}x^2 + 2f_{xy}xy + f_{yy}y^2$$

222. Taylor's theorem. By the following method, Taylor's theorem may be extended to functions of two or more variables.

Suppose that $f(x, y)$ and its partial derivatives to those of order n are continuous in the rectangle R bounded by $x = a$, $x = a + h$, $y = b$, $y = b + k$. It is required to develop $f(a + h, b + k)$ in powers of h, k.

Let t denote a variable whose range is from 0 to 1, and set

$$F(t) = f(a + ht, b + kt) \tag{1}$$

From the continuity of $f(x, y)$ and its partial derivatives it follows that $F(t)$, $F'(t)$, $\cdots$, $F^{(n)}(t)$ are continuous in the t-interval $(0, 1)$. Hence we may expand $F(t)$ in powers of t to the nth by Maclaurin's formula, § 188 (9), and then set $t = 1$; which gives

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \cdots + \frac{F^{(n-1)}(0)}{(n-1)!} + \frac{F^{(n)}(\theta)}{n!} \quad 0 < \theta < 1 \tag{2}$$

By (1), we have $F(1) = f(a + h, b + k)$ and $F(0) = f(a, b)$.

Differentiating (1) with respect to t, and then setting $t = 0$, gives

$$F'(0) = \frac{\partial}{\partial a} f(a, b)h + \frac{\partial}{\partial b} f(a, b)k = \left(\frac{\partial}{\partial a} h + \frac{\partial}{\partial b} k\right) f(a, b) \tag{3}$$

Dealing similarly with $F'(t)$ gives

$$F''(0) = \frac{\partial^2}{\partial a^2} f(a, b)h^2 + 2\frac{\partial^2}{\partial a\, \partial b} f(a, b)hk + \frac{\partial^2}{\partial b^2} f(a, b)k^2 \tag{4}$$

which may be expressed symbolically in the form

$$F''(0) = \left(\frac{\partial}{\partial a} h + \frac{\partial}{\partial b} k\right)^2 f(a, b) \tag{5}$$

and so on. Hence, by substitution in (2),

$$f(a + h, b + k) = f(a, b) + \left(\frac{\partial}{\partial a} h + \frac{\partial}{\partial b} k\right) f(a, b) + \cdots$$
$$+ \frac{1}{(n-1)!}\left(\frac{\partial}{\partial a} h + \frac{\partial}{\partial b} k\right)^{n-1} f(a, b) + \frac{1}{n!}\left(\frac{\partial}{\partial a} h + \frac{\partial}{\partial b} k\right)^{n} f(a + \theta h, b + \theta k) \tag{6}$$

the terms after the first being homogeneous polynomials of degrees 1, 2, $\cdots$ in h, k.

Example. The curve $x^2y^3 = 4$ passes through the point $O'(2, 1)$. By aid of Taylor's theorem, find the equation of the curve referred to axes $O'x'$, $O'y'$ parallel to Ox, Oy. See page 237, Ex. 12.

223. Mean value theorem for $f(x, y)$. 1. In the formula § 222 (6), take $n = 1$, and transpose the term $f(a, b)$; we get

$$f(a + h, b + k) - f(a, b) = f_x(a + \theta h, b + \theta k)h + f_y(a + \theta h, b + \theta k)k$$

or, replacing $a + h$, $b + k$ by x, y, and $a + \theta h$, $b + \theta k$ by x_1, y_1,

$$f(x, y) - f(a, b) = f_x(x_1, y_1)(x - a) + f_y(x_1, y_1)(y - b) \quad (1)$$

where (x_1, y_1) is some point on the line segment joining the points (x, y) and (a, b).

In like manner, if $f(x, y, z)$, f_x, f_y, f_z are continuous in the parallelepiped bounded by planes taken through the points (a, b, c) and (x, y, z) parallel to the yz-, zx-, and xy-planes, there is a point (x_1, y_1, z_1) on the line segment joining (a, b, c) and (x, y, z) such that

$$f(x, y, z) - f(a, b, c) = f_x(x_1, y_1, z_1)(x - a) + f_y(x_1, y_1, z_1)(y - b) + f_z(x_1, y_1, z_1)(z - c) \quad (2)$$

2. It follows from (1) that if f_x and f_y are 0 throughout the rectangle R of § 222, then $f(x, y)$ is a constant in R.

Hence two functions $\phi(x)$ and $\psi(x)$ differ by a constant only if $\phi_x = \psi_x$ and $\phi_y = \psi_y$ throughout R. For if we set $f(x) = \phi(x) - \psi(x)$, then f_x and f_y are 0 throughout R.

The corresponding theorems for functions of x, y, z follow from (2).

IMPLICIT FUNCTIONS

224. Definition of an implicit function. Let $f(x, y) = 0$ be a given equation in x, y, and let (a, b) be a point of its graph, so that $f(a, b)$ is 0. Suppose that a rectangle R bounded by lines

$$x = a - k \qquad x = a + k \qquad y = b - l \qquad y = b + l$$

can be found such that the portion of the graph of $f(x, y) = 0$ within R is a curve arc C which is met by each parallel to Oy

that crosses R in one and but one point. Then to each value x' of x between $a - k$ and $a + k$ corresponds a single value y' of y between $b - l$ and $b + l$ such that $f(x', y')$ is 0. Hence in R the equation $f(x, y) = 0$ defines y as a one-valued function of x, § 14. Let $y = \phi(x)$ denote this function. It is said to be *defined implicitly* by the equation $f(x, y) = 0$ and the point (a, b). It may also be called the solution of $f(x, y) = 0$ for y in terms of x at (a, b) and in its neighborhood; for $f[x, \phi(x)] \equiv 0$.

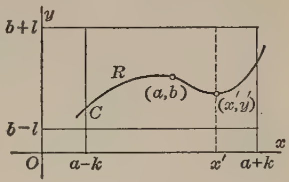

FIG. 118.

225. Theorem. *Let $f(x, y)$, f_x, f_y be continuous at the point $P(a, b)$ and in its neighborhood. If f is 0 and $f_y \neq 0$ at P, the equation $f(x, y) = 0$ has a solution $y = \phi(x)$ at P and in its neighborhood, and but one such solution.*

Furthermore $\phi'(x) = - f_x(x, y)/f_y(x, y)$; and $\phi'(x)$ is continuous at P and in its neighborhood.

1. By hypothesis, f, f_x, f_y are continuous in some region S about P, § 214, 6. We are to prove that in S there exists a rectangle R, of the kind described in § 224, in which the graph of $f(x, y) = 0$ is met by parallels to Oy in single points.

First consider the y-function $f(a, y)$. It is continuous in S. It vanishes at $y = b$ (since $f(a, b) = 0$), and there changes sign (since its derivative at $y = b$, namely $f_y(a, b)$, is not 0). Hence, § 31, a positive number l can be found such that $f(a, b - l)$ and $f(a, b + l)$ have opposite signs.

Next consider the x-functions $f(x, b - l)$ and $f(x, b + l)$. They are continuous in S and therefore, for values of x sufficiently near a, they have the same signs as $f(a, b - l)$ and $f(a, b + l)$ respectively (p. 22, footnote). Hence a positive number k can be found such that

$$\operatorname{sgn} f(x, b - l) = - \operatorname{sgn} f(x, b + l), \text{ when } |x - a| < k \qquad (1)$$

Moreover $f_y(x, y)$ is continuous in S. Therefore, since it is different from 0 at (a, b), it is different from 0 at all points (x, y) sufficiently near (a, b), that is, we may and shall take k, l so that they also satisfy the condition

$$f_y \neq 0 \text{ when } |x-a| < k, |y-b| < l \tag{2}$$

Finally, let x' be any value of x between $a-k$ and $a+k$, and consider the y-function $f(x', y)$. By (1), it has opposite signs for $y = b - l$ and $y = b + l$. It therefore vanishes for some value y' of y between $b - l$ and $b + l$, § 18, 2. And there can be but one such value y' since, were there more than one, $f_y(x', y)$ would vanish for some value of y between them, § 96, which is impossible, by (2).

We have thus proved that in the rectangle R bounded by the lines $x = a - k$, $x = a + k$, $y = b - l$, $y = b + l$, to each value x' of x corresponds a single value y' of y such that $f(x', y')$ is 0. Hence in R the equation $f(x, y) = 0$ defines y as a one-valued function of x. This function $y = \phi(x)$ is the solution, and the only solution, of the equation $f(x, y) = 0$ for y in terms of x in R.

2. It remains to prove that

$$\phi'(x) = -f_x(x, y)/f_y(x, y)$$

Let (x', y') and $(x' + \Delta x, y' + \Delta y)$ be points in R such that $f(x', y') = 0$ and $f(x' + \Delta x, y' + \Delta y) = 0$, and therefore such that $y' = \phi(x')$, and $y' + \Delta y = \phi(x' + \Delta x)$.

Since $f(x' + \Delta x, y' + \Delta y) - f(x', y') = 0$, it follows from § 223 that

$$f_x(x' + \theta\,\Delta x, y' + \theta\,\Delta y)\Delta x + f_y(x' + \theta\,\Delta x, y' + \theta\,\Delta y)\Delta y = 0$$

Therefore, since f_x is continuous, and f_y continuous and $\neq 0$, in R,

$$\frac{\Delta y}{\Delta x} = -\frac{f_x(x' + \theta\,\Delta x, y' + \theta\,\Delta y)}{f_y(x' + \theta\,\Delta x, y' + \theta\,\Delta y)} \quad \therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{f_x(x', y')}{f_y(x', y')}$$

But $\lim\limits_{\Delta x \to 0} (\Delta y/\Delta x) = \phi'(x')$. Hence $\phi'(x') = -f_x(x', y')/f_y(x', y')$.

Moreover $\phi'(x)$ is continuous in R; for f_x, f_y, and $\phi(x)$ are continuous, and $\phi'(x) = -f_x[x, \phi(x)]/f_y[x, \phi(x)]$.

By similar reasoning we can prove the following more general theorem:

226. Theorem. *Let $f(u, x, y, \cdots)$, f_u, f_x, $f_y \cdots$ be continuous at the point $P(u_1, a, b, \cdots)$ and in its neighborhood. If f is 0 and $f_u \neq 0$ at P, then in a certain region*[1] *R about P the equation $f(u, x, y, \cdots) = 0$ defines u as a one-valued function of $x, y, \cdots$. This function $u = \phi(x, y, \cdots)$ is the solution and the only solution of $f(u, x, y, \cdots) = 0$ for u in terms of $x, y, \cdots$ in the region R.*

Furthermore, $\phi_x(x, y, \cdots) = - f_x(u, x, y, \cdots)/f_u(u, x, y, \cdots)$, and so on; and $\phi_x, \phi_y, \cdots$ are continuous in R.

227. Differentiation of implicit functions. 1. The function $y = \phi(x)$ defined by $f(x, y) = 0$, at any point $P(x, y)$ where $f_y \neq 0$, has finite derivatives $dy/dx, \cdots, d^ny/dx^n$ at P if f has finite partial derivatives to those of order n at P.

For it has already been proved that $dy/dx = - f_x/f_y$. Hence

$$\frac{d^2y}{dx^2} = -\left[\frac{\partial}{\partial x}\frac{f_x}{f_y} + \frac{\partial}{\partial y}\frac{f_x}{f_y}\cdot\frac{dy}{dx}\right] = -\frac{f_{xx}(f_y)^2 - 2f_{xy}f_xf_y + f_{yy}(f_x)^2}{(f_y)^3} \qquad (1)$$

and repetitions of this process will give d^ny/dx^n in the form of a fraction whose numerator is an integral expression in partial derivatives of f to those of order n and whose denominator is some power of f_y.

Having proved that $dy/dx, d^2y/dx^2, \cdots$ exist under the conditions just indicated, we may conclude, as in § 42, 2, that they satisfy the equations got from $f(x, y) = 0$ by differentiating repeatedly with respect to x, regarding y as a function of x.

Thus, differentiating $f(x, y) = 0$ gives $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx} = 0 \qquad (2)$

Differentiating (2) gives

$$\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\,\partial y}\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}\frac{d^2y}{dx^2} = 0 \qquad (3)$$

and so on. In (3) substitute $dy/dx = - f_x/f_y$, given by (2), and then solve for d^2y/dx^2. We thus obtain (1).

2. Similarly, in the equation $f(x, y, z) = 0$, regard x, y as the independent variables, and z as standing for the function

[1] The region R consisting of all points $(u, x, y, \cdots)$ for which $|x - a| < k$, $|y - b| < k, \cdots$, and $|u - u_1| < l$, where k, l denote certain positive numbers.

$z = \phi(x, y)$ defined by the equation at any point $P(x, y, z)$ where $f_z \neq 0$. Then $\partial z/\partial x$, $\partial z/\partial y$, $\cdots$ stand for ϕ_x, ϕ_y, $\cdots$, and z has partial derivatives of order n at P if f has finite partial derivatives to those of order n at P. Furthermore, equations which determine $\partial z/\partial x$, $\partial z/\partial y$, $\cdots$ may be got from $f(x, y, z) = 0$ by differentiating repeatedly with respect to x (regarding z as a function of x, and y as a constant), and with respect to y (regarding z as a function of y, and x as a constant).

Thus,
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0,$$
which give
$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}, \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$$

Example 1. If $xyz + 2x + y = 2$ (1), find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, and $\frac{\partial^2 z}{\partial x\,\partial y}$ at $(1, 2, -1)$.

Differentiating (1) with respect to x,
$$yz + 2 + xy\frac{\partial z}{\partial x} = 0 \qquad (2)$$
Differentiating (1) with respect to y,
$$xz + 1 + xy\frac{\partial z}{\partial y} = 0 \qquad (3)$$
Differentiating (2) with respect to y,
$$z + y\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial x} + xy\frac{\partial^2 z}{\partial x\,\partial y} = 0 \qquad (4)$$

Setting $x, y, z = 1, 2, -1$ and solving, we find
$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0, \quad \frac{\partial^2 z}{\partial x\,\partial y} = \frac{1}{2}$$

Example 2. If $z^2 + zx + y = 0$, find $\partial^2 z/\partial x^2$ in terms of x, y, z.

3. In the equation $f(x, y, z) = 0$, any two of x, y, z may be taken as the independent variables. If y, z be so taken, $f(x, y, z) = 0$ defines x as a function of y, z at every point where f is 0 and $f_x \neq 0$; and all that has been said regarding the partial derivatives of z with respect to x and y is true of the partial derivatives of x with respect to y and z. Similarly, if x, z are taken as the independent variables.

Example 3. If $x^3 + xz - 2y = 0$, find $\partial^2 x/\partial y^2$ in terms of x, y, z.

Example 4. If $f(x, y, z) = 0$, prove that $\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial z} = 1$; also that
$$\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x} = -1.$$

228. Expression of an implicit function by Taylor's series. 1. Let y denote the function $y = \phi(x)$ defined by $f(x, y) = 0$ at any given point (a, b) where f is 0 and $f_y \neq 0$. Then, representing $b = \phi(a)$ by y_a, $\phi'(a)$ by $(dy/dx)_a$, and so on, we have by § 188 (7),

$$y = y_a + \left(\frac{dy}{dx}\right)_a \frac{(x-a)}{1} + \left(\frac{d^2y}{dx^2}\right)_a \frac{(x-a)^2}{2!} + \cdots$$
$$+ \left(\frac{d^{n-1}y}{dx^{n-1}}\right)_a \frac{(x-a)^{n-1}}{(n-1)!} + \left(\frac{d^ny}{dx^n}\right)_{x_1} \frac{(x-a)^n}{n!} \qquad (1)$$

where x_1 denotes some number between a and x.

The expression holds good for every value of x between which and a the derivatives $dy/dx, \cdots d^ny/dx^n$ are finite. For values of x sufficiently near a, we may use two or three terms of (1) to find the corresponding values of y approximately, § 193.

2. Similarly, by aid of Taylor's series for functions of two variables, § 222, we can find an expression for the function $z = \phi(x, y)$ defined by $f(x, y, z) = 0$ at a point (a, b, c) where f is 0 and $f_z \neq 0$, this expression being one in positive integral powers of $x - a$, $y - b$.

EXAMPLE 1. Find to four terms the solution of $x^2 + y^2 = 5$ for y at the point (2, 1).

By successive differentiations of $x^2 + y^2 - 5 = 0$, we obtain

$$x + y\frac{dy}{dx} = 0, \qquad 1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0, \qquad 3\frac{dy}{dx}\frac{d^2y}{dx^2} + y\frac{d^3y}{dx^3} = 0$$

Substituting $x, y = 2, 1$, we find $\dfrac{dy}{dx} = -2$, $\dfrac{d^2y}{dx^2} = -5$, $\dfrac{d^3y}{dx^3} = -30$.

Hence, $\quad y = 1 - 2(x-2) - 5\dfrac{(x-2)^2}{2} - 30\dfrac{(x-2)^3}{6} + \cdots$

EXAMPLE 2. Find to three terms the solution of $y^3 + y^2 + y = x$ for y at the point (0, 0).

229. Functions defined by simultaneous equations. Functional determinants. Let F_1 and F_2 denote functions of u, v. The determinant

$$J = \begin{vmatrix} \dfrac{\partial F_1}{\partial u} & \dfrac{\partial F_1}{\partial v} \\ \dfrac{\partial F_2}{\partial u} & \dfrac{\partial F_2}{\partial v} \end{vmatrix} = \frac{\partial F_1}{\partial u}\frac{\partial F_2}{\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial F_2}{\partial u} \tag{1}$$

is called the *functional determinant* or *Jacobian* of F_1, F_2 with respect to u, v.

In general, if $F_1, F_2, \cdots, F_n$ are functions of $u_1, u_2, \cdots, u_n$, (and it may be of other variables also), the functional determinant of $F_1, F_2, \cdots, F_n$ with respect to $u_1, u_2, \cdots u_n$ is

$$J = \begin{vmatrix} \dfrac{\partial F_1}{\partial u_1} & \dfrac{\partial F_1}{\partial u_2} & \cdots & \dfrac{\partial F_1}{\partial u_n} \\ \dfrac{\partial F_2}{\partial u_1} & \dfrac{\partial F_2}{\partial u_2} & \cdots & \dfrac{\partial F_2}{\partial u_n} \\ \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial F_n}{\partial u_1} & \dfrac{\partial F_n}{\partial u_2} & \cdots & \dfrac{\partial F_n}{\partial u_n} \end{vmatrix} \tag{2}$$

Example. If $F_1 = a_1u + b_1v + c_1$, $F_2 = a_2u + b_2v + c_2$, then $\partial F_1/\partial u = a_1, \cdots$; hence

$$J = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$$

Show that the equations $F_1 = 0$, $F_2 = 0$ have a solution, and but one, for u, v in terms of $a_1, b_1, \cdots c_2$, when, and only when, $J \neq 0$. Here J is the determinant Δ of § 202.

230. Theorem. *Let $F_1, F_2, \cdots, F_n$ be functions of the n variables $u, v, \cdots$, and of any number of other variables $x, y, \cdots$; let J denote the functional determinant of $F_1, F_2, \cdots$ with respect to $u, v, \cdots$; and let the functions $F_1, F_2, \cdots, F_n$ and their partial derivatives of the first order be continuous at the point P $(u_1, v_1, \cdots;\ a, b \cdots)$ and in its neighborhood.*

Suppose that at P the functions $F_1, F_2, \cdots, F_n$ all have the value 0, but $J \neq 0$. Then in some region R about P the equations

$$F_1 = 0, F_2 = 0, \cdots, F_n = 0$$

define $u, v, \cdots$ *as one-valued functions of* $x, y, \cdots$; *that is, a set of* n *functions*

$$u = \phi_1(x, y, \cdots), \qquad v = \phi_2(x, y, \cdots), \qquad w = \phi_3(x, y, \cdots), \cdots$$

exists, and but one such set, which satisfy the equations $F_1 = 0$, $F_2 = 0, \cdots, F_n = 0$ *identically in the region* R, *and have the values* $u_1, v_1, \cdots$ *when* $x, y, \cdots = a, b, \cdots$.

Moreover $\phi_1, \phi_2, \cdots$ *have continuous partial derivatives of the first order at* P *and in its neighborhood.*

1. First consider the case of the two functions

$$F_1(u, v, x) \quad F_2(u, v, x), \quad \text{where} \quad J = \frac{\partial F_1}{\partial u}\frac{\partial F_2}{\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial F_2}{\partial u}.$$

Since $J \neq 0$ at P, at least one of $\partial F_2/\partial v$, $\partial F_2/\partial u$ is not 0 at P. Suppose that $\partial F_2/\partial v \neq 0$.

Then since F_2 is 0 and $\partial F_2/\partial v \neq 0$ at P, the equation $F_2 = 0$ has a solution for v in terms of u, x at P and in its neighborhood, § 226; let this solution be

$$v = V(u, x) \tag{1}$$

so that

$$F_2[u, V(u, x), x] \equiv 0 \tag{2}$$

Substitute (1) in F_1, and represent the resulting function of u, x by $\Phi(u, x)$; so that

$$F_1[u, V(u, x), x] \equiv \Phi(u, x) \tag{3}$$

Since F_1 is 0 at P, so also is Φ; and, as will be shown immediately, $\Phi_u \neq 0$ at P. Hence the equation $\Phi = 0$ has a solution $u = \phi(x)$ at P and in its neighborhood, § 225. Substituting this in (1), and calling the result $v = \psi(x)$, we have

$$u = \phi(x) \qquad v = \psi(x) \tag{4}$$

which is the solution of $F_1 = 0$, $F_2 = 0$ for u, v at P and in its neighborhood.

We have assumed that $\Phi_u \neq 0$. To prove this, differentiate the identities (2) and (3) with respect to u. We obtain

$$\frac{\partial \Phi}{\partial u} \equiv \frac{\partial F_1}{\partial u} + \frac{\partial F_1}{\partial v}\frac{\partial V}{\partial u} \qquad 0 \equiv \frac{\partial F_2}{\partial u} + \frac{\partial F_2}{\partial v}\frac{\partial V}{\partial u}$$

Eliminating $\dfrac{\partial V}{\partial u}$ between these equations, we get

$$\frac{\partial F_2}{\partial u}\frac{\partial \Phi}{\partial u} \equiv J. \quad \text{But } \frac{\partial F_2}{\partial v}, J \neq 0. \quad \text{Hence } \frac{\partial \Phi}{\partial u} \neq 0.$$

2. We next prove that if the theorem is true for $n - 1$ of the functions $F_1, F_2, \cdots, F_n$ with respect to $n - 1$ of the variables $u, v, \cdots$, it is

true for all n of $F_1, F_2, \cdots, F_n$ with respect to all n of $u, v, \cdots$. It can then be inferred from 1. that the theorem is true when $n = 3$, hence when $n = 4$, and so on.

Representing the cofactors of the elements of the first column of J by $J_1, J_2, \cdots J_n$, we have, § 201, 1.,

$$J = \frac{\partial F_1}{\partial u} J_1 + \frac{\partial F_2}{\partial u} J_2 + \cdots + \frac{\partial F_n}{\partial u} J_n$$

Hence, since $J \neq 0$ at P, at least one of $J_1, J_2, \cdots, J_n$ is not 0 at P. Suppose $J_1 \neq 0$. Then, since J_1 is the functional determinant of $F_2, \cdots, F_n$ with respect to $v, w, \cdots$, the equations $F_2 = 0, \cdots, F_n = 0$ have, by hypothesis, a solution for $v, w, \cdots$ in terms of $u, x, y, \cdots$ at P and in its neighborhood. Let this solution be

$$v = V(u, x, y, \cdots), \qquad w = W(u, x, y, \cdots), \cdots \tag{5}$$

Substitute (5) in F_1 and represent the resulting function of $u, x, \cdots$ by Φ. Then

$$\left.\begin{aligned} & F_1(u, V, W, \cdots, x, y, \cdots) \equiv \Phi(u, x, y, \cdots) \\ \text{Also} \quad & F_2(u, V, W, \cdots) \equiv 0, \quad F_3(u, V, W, \cdots) \equiv 0, \cdots \end{aligned}\right\} \tag{6}$$

At P, Φ is 0 but $\Phi_u \neq 0$. For differentiating the identities (6) with respect to u,

$$\frac{\partial \Phi}{\partial u} \equiv \frac{\partial F_1}{\partial u} + \frac{\partial F_1}{\partial v}\frac{\partial V}{\partial u} + \frac{\partial F_1}{\partial w}\frac{\partial W}{\partial u} + \cdots,$$

$$0 \equiv \frac{\partial F_2}{\partial u} + \frac{\partial F_2}{\partial v}\frac{\partial V}{\partial u} + \frac{\partial F_2}{\partial w}\frac{\partial W}{\partial u} + \cdots,$$

$$0 \equiv \frac{\partial F_3}{\partial u} + \frac{\partial F_3}{\partial v}\frac{\partial V}{\partial u} + \frac{\partial F_3}{\partial w}\frac{\partial W}{\partial u} + \cdots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Multiply the first of these identities by J_1, the second by J_2, and so on, and add the results. We get, by the theorems of § 201,

$$J_1 \frac{\partial \Phi}{\partial u} \equiv J. \quad \text{But } J, J_1, \neq 0. \quad \text{Hence } \frac{\partial \Phi}{\partial u} \neq 0$$

Hence the equation $\Phi(u, x, y, \cdots) = 0$ has a solution $u = \phi_1(x, y, \cdots)$, and but one, at P and in its neighborhood, § 226. Substituting this in (5), we have finally

$$u = \phi_1(x, y, \cdots), \quad v = \phi_2(x, y, \cdots), \quad w = \phi_3(x, y, \cdots), \cdots \tag{7}$$

which is the solution of $F_1 = 0, \cdots F_n = 0$ for $u, v, \cdots$ at P and in its neighborhood.

That the functions $\phi_1, \phi_2, \cdots$ have continuous partial derivatives of the first order at P and in its neighborhood follows from the last clause of the theorem of § 226.

In applying this theorem to a given set of n equations in $m(> n)$ variables, any m of the variables for which the corresponding functional determinant J is not 0 may be taken as dependent, the rest being independent.

231. Differentiation of functions defined by simultaneous equations. By § 230, the equations $F_1(u, v, x, y) = 0$, $F_2(u, v, x, y) = 0$ define u and v as functions of x, y at every point P where $F_1 = 0$, $F_2 = 0$ and $J \neq 0$; and the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist at P. These partial derivatives and those of higher order may be found by a procedure analogous to that of § 227.

Differentiating $F_1(u, v, x, y) = 0$, $F_2(u, v, x, y) = 0$ with respect to x

$$\frac{\partial F_1}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F_1}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial F_1}{\partial x} = 0 \qquad \frac{\partial F_2}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F_2}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial F_2}{\partial x} = 0 \tag{1}$$

Since $J \neq 0$, we can solve this pair of equations for $\partial u/\partial x$, $\partial v/\partial x$:

$$\frac{\partial u}{\partial x} = \left(\frac{\partial F_1}{\partial v}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial v}\right)\Big/ J \qquad \frac{\partial v}{\partial x} = \left(\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial u} - \frac{\partial F_1}{\partial u}\frac{\partial F_2}{\partial x}\right)\Big/ J \tag{2}$$

The corresponding formulas for $\partial u/\partial y$, $\partial v/\partial y$ may be got by replacing x by y in (2).

In the pair of equations got by differentiating the pair (1) with respect to x (or y) the determinant of the coefficients of $\partial^2 u/\partial x^2$ and $\partial^2 v/\partial x^2$ (or $\partial^2 u/\partial x\,\partial y$ and $\partial^2 v/\partial x\,\partial y$) is again J; and so on. It therefore follows, by an obvious extension of the reasoning in § 227, that if $J \neq 0$, and the partial derivatives of F_1, F_2 to those of order n exist, then the partial derivatives of u and v to those of order n also exist and are determined by the equations got by repeated differentiations of $F_1 = 0$, $F_2 = 0$ with respect to x and y.

EXAMPLE. Show that the pair of equations $y^2 - z - x = 0$, $z^2 + y - 2x = 0$ has a solution for y and z in terms of x at O and in its neighborhood, and express this solution by Taylor series to two terms each.

$$2y\frac{dy}{dx} - \frac{dz}{dx} - 1 = 0$$

$$\frac{dy}{dx} + 2z\frac{dz}{dx} - 2 = 0 \qquad \text{Hence, at } O,\ J = 1,\ \frac{dy}{dx} = 2,\ \frac{dz}{dx} = -1$$

$$2y\frac{d^2y}{dx^2} - \frac{d^2z}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$$

$$\frac{d^2y}{dx^2} + 2z\frac{d^2z}{dx^2} + 2\left(\frac{dz}{dx}\right)^2 = 0 \quad \text{Hence, at } O, \ \frac{d^2y}{dx^2} = -2, \ \frac{d^2z}{dx^2} = 8$$

Therefore, $\qquad y = 2x - x^2 + \cdots \qquad z = -x + 4x^2 + \cdots$

232. Interdependence of functions. Two functions of x, y may be functions one of the other. Thus if $u = (x + y)/y$ and $v = (y/x)^2$, then $v = 1/(u - 1)^2$.

Let the functions

$$u = \phi(x, y) \quad (1) \qquad\qquad v = \psi(x, y) \quad (2)$$

and their partial derivatives of the first order be continuous at the point P and in some region R about P, and suppose that at least one of these partial derivatives, say ϕ_y, is not 0 in R.

1. *If v be a function of u, say $v = f(u)$, then in R*

$$J = \phi_x\psi_y - \phi_y\psi_x = 0$$

For, differentiating $v = f(u)$ with respect to x and y,

$$\psi_x - f'(u)\phi_x = 0 \qquad\qquad \psi_y - f'(u)\phi_y = 0$$

Hence, eliminating $f'(u)$, $\quad \phi_x\psi_y - \phi_y\psi_x = 0$

2. *Conversely, if $J = 0$ in R, then v is a function of u in R.*

Since $\phi_y \neq 0$ in R, the equation (1) has a solution for y in terms of u, x at any point Q of R, say $y = Y(u, x)$ (3). It is to be proved that, when $J = 0$, then $\psi[x, Y(u, x)]$ is a function of u only, and not of x.

This is equivalent to showing that when u and x are taken as the independent variables in (1) and (2), and $J = 0$, then $\partial v/\partial x = 0$.

But when u, x are the independent variables, differentiating (1) and (2) with respect to x gives

$$0 = \phi_x + \phi_y\frac{\partial y}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial x} = \psi_x + \psi_y\frac{\partial y}{\partial x}$$

Eliminating $\dfrac{\partial y}{\partial x}$ between these equations gives

$$\frac{\partial v}{\partial x}\phi_y = -J. \quad \text{But } J = 0 \text{ and } \phi_y \neq 0. \quad \text{Hence, } \frac{\partial v}{\partial x} = 0$$

3. Under like conditions, and by similar reasoning, it can be shown that if

$$u = \phi(x, y, z) \qquad v = \psi(x, y, z) \qquad w = \chi(x, y, z)$$

the necessary and sufficient condition that one of the functions u, v, w be a function of the other two is that the functional determinant of ϕ, ψ, χ with respect to x, y, z be identically 0.

The theorem admits of extension to the general case of n functions of n independent variables.

EXERCISE XLV

1. Find the indicated solutions of the following equations to the terms involving the second powers of the independent variable or variables.

(1) $x^2 - y^2 + x - 2y = 0$ for y in terms of x, and x in terms of y, at $(0, 0)$.

(2) $y^3 - 4y + 3x = 0$ for y in terms of x, at $(1, 1)$.

(3) $z^2 + yz + z - y + x = 0$ for z in terms of x and y, at $(0, 0, 0)$.

(4) $y^2 + 3z^2 - 4x = 0$, $z + 2y + x = 0$ for y and z in terms of x, at $(1, -1, 1)$.

2. Show that in $u = xy$, $x + 2y - 3z = 4$, any two of x, y, z, u may be chosen as the independent variables. Find $\partial u/\partial x$ for each of the choices: x, y; x, z; x, u.

3. If $u^3 + v^3 + x^3 - 3y = 0$ and $u^2 + v^2 + y^2 + 2x = 0$, find $\partial u/\partial x$, $\partial v/\partial x$.

4. If $z = uv$, and $u^2 - v + x = 0$, $u + v^2 - y = 0$, find $\partial z/\partial x$; $\partial z/\partial y$.

5. If $f(ux, uy) = 0$, find $\partial u/\partial x$, $\partial u/\partial y$.

6. If u is a function of $x - y$, $y - z$, $z - x$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

7. Show that if $F(x, y, z, t) = 0$, then $\dfrac{\partial x}{\partial y} \cdot \dfrac{\partial y}{\partial z} \cdot \dfrac{\partial z}{\partial t} \cdot \dfrac{\partial t}{\partial x} = 1$.

8. If $f_1(u, v) = 0$, $f_2(v, y) = 0$, $f_3(y, x) = 0$, find du/dx.

9. If $x = \phi(u, v)$ and $y = \psi(u, v)$, show that

$$\frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} = 1, \quad \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = 0, \text{ and find } \frac{\partial u}{\partial x}$$

10. If $x = r\cos\theta$, $y = r\sin\theta$, show that $\partial x/\partial r = \partial r/\partial x$.

11. Let u and v be functions of x, y such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Show that if we set $x = r\cos\theta$, $y = r\sin\theta$, we find

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

12. In $f(x, y)$ set $y = tx$, and represent $f(x, xt)$ by $\phi(x, t)$. Prove that at a point where $f_x = 0$, $f_y = 0$, we have $\phi_x = 0$, $\phi_t = 0$.

13. By expanding $f(x, y)$ in powers of $x - x_1$, $y - y_1$, at the point (x_1, y_1), § 222, show that if the line $y - y_1 = m(x - x_1)$ touches the curve $f(x, y) = 0$ at (x_1, y_1), then $f[x, y_1 + m(x - x_1)]$ is divisible by $(x - x_1)^2$.

XXIV. APPLICATIONS OF PARTIAL DIFFERENTIATION

SURFACES. CURVES IN SPACE

233. Tangent plane to a surface. Let S be a given surface, and $f(x, y, z) = 0$ its equation; also let $P(x', y', z')$ be a given point of S and $Q(x' + \Delta x,\ y' + \Delta y,\ z' + \Delta z)$ any adjacent point of the surface, so that $f(x', y', z') = 0$, and $f(x' + \Delta x,\ y' + \Delta y,\ z' + \Delta z) = 0$.

Then by the mean value theorem § 223 (2),

$$f_x(x_1, y_1, z_1)\Delta x + f_y(x_1, y_1, z_1)\Delta y + f_z(x_1, y_1, z_1)\Delta z = 0 \quad (1)$$

where (x_1, y_1, z_1) is some point between P and Q on the line segment PQ.

The direction ratios of the line PQ are $\Delta x : \Delta y : \Delta z$. Hence, § 212, the equation (1) shows that the line PQ is perpendicular to the line L whose direction ratios are

$$f_x(x_1, y_1, z_1) : f_y(x_1, y_1, z_1) : f_z(x_1, y_1, z_1)$$

Suppose Q to move on S into coincidence with P. Then the line PQ becomes a line tangent to S at P, the ratios $\Delta x : \Delta y : \Delta z$ become the direction ratios of this tangent line, and L becomes a line N perpendicular to this tangent and having the direction ratios

$$f_x(x', y', z') : f_y(x', y', z') : f_z(x', y', z') \quad (2)$$

Hence the line N which has the direction ratios (2) is perpendicular to every line which touches S at P. Therefore all these tangent lines lie in the plane perpendicular to N through P. This plane is defined as the *tangent plane* to S at P, and N is called the *normal* to S at P.

Therefore, by § 213, the equation of the tangent plane to S at P is

$$\frac{\partial f}{\partial x'}(x - x') + \frac{\partial f}{\partial y'}(y - y') + \frac{\partial f}{\partial z'}(z - z') = 0 \qquad (3)$$

and, § 210, the equations of the normal are

$$\frac{x - x'}{\partial f/\partial x'} = \frac{y - y'}{\partial f/\partial y'} = \frac{z - z'}{\partial f/\partial z'} \qquad (4)$$

When the equation $f = 0$ is of the form $z - \phi(x, y) = 0$, then (3) becomes

$$z - z' = \frac{\partial z'}{\partial x'}(x - x') + \frac{\partial z'}{\partial y'}(y - y') \qquad (5)$$

Observe that $\partial z'/\partial x'$ and $\partial z'/\partial y'$ are the slopes at P of the curves in which S is cut by the planes $y = y'$ and $x = x'$.

In deriving (3) we have assumed that f and its partial derivatives f_x, f_y, f_z are continuous at P and in its neighborhood, that is, in a region of the kind described in § 214, 6 in the space about P; also that at least one of f_x, f_y, f_z is not 0 at P. A point where these conditions are not satisfied is called a *singular point* of S; for example, the vertex O of the conical surface $x^2 + y^2 - z^2 = 0$. Obviously (3) is illusory at a point where f_x, f_y, f_z are all 0.

234. Tangents to space curves. 1. Consider the equations

$$x = f(t) \qquad y = g(t) \qquad z = h(t) \qquad (1)$$

where f, g, h denote functions having continuous derivatives which do not vanish simultaneously. To each value of t in the t-interval for which f, g, h are defined corresponds a single set of values of x, y, z and therefore a single point P. When t varies continuously, P traces a curve C. We call (1) the equations of C in terms of the parameter t. Compare § 56.

Let $P(x', y', z')$, corresponding to $t = t'$, be a given point of C; and let $Q(x' + \Delta x, y' + \Delta y, z' + \Delta z)$, corresponding to $t = t' + \Delta t$, be any adjacent point of C. The direction ratios of the line PQ are $\Delta x : \Delta y : \Delta z$. Suppose that $\Delta t \to 0$ and therefore that $Q \to P$ along C. In the limit, PQ be-

comes the tangent to C at P, and $\Delta x : \Delta y : \Delta z$ become the direction ratios of this tangent. But

$$\Delta x : \Delta y : \Delta z = \frac{\Delta x}{\Delta t} : \frac{\Delta y}{\Delta t} : \frac{\Delta z}{\Delta t} \rightarrow \frac{dx}{dt} : \frac{dy}{dt} : \frac{dz}{dt} \tag{2}$$

Therefore the equations of the tangent to C at P are

$$\frac{x - x'}{f'(t')} = \frac{y - y'}{g'(t')} = \frac{z - z'}{h'(t')} \tag{3}$$

EXAMPLE 1. The equations $x = a \cos \theta$, $y = a \sin \theta$, $z = k\theta$ represent a *helix*. Show that the curve has the form of a spiral surrounding a right circular cylinder of radius a and with its axis along Oz. Show also that the curve meets the elements of the cylinder at a constant angle.

2. A space curve C may also be represented by the equations $F_1 = 0$, $F_2 = 0$ of two surfaces of which it is the intersection. By § 220, the equations $F_1 = 0$, $F_2 = 0$ have one and but one solution for y and z in terms of x at any point P of C where the functional determinant J of F_1, F_2 with respect to y, z is not 0. Call this solution

$$y = f(x) \qquad z = \phi(x) \tag{4}$$

The equations (4) are equivalent to (1) when $t = x$ and represent C in the neighborhood of P in terms of the parameter x. By setting $t = x$ in (2), we obtain for the direction ratios of the tangent at P,

$$1 : \frac{dy}{dx} : \frac{dz}{dx} \quad \text{or} \quad dx : dy : dz \tag{5}$$

EXAMPLE 2. Find the direction ratios of the tangent to the circle $x^2 + y^2 + z^2 - 3x = 0$, $2x - 3y + 5z - 4 = 0$ at $(1, 1, 1)$.

Differentiating, $(2x - 3)\,dx + 2y\,dy + 2z\,dz = 0$

$$2\,dx - 3\,dy + 5\,dz = 0$$

Hence at $(1, 1, 1)$

$$-dx + 2\,dy + 2\,dz = 0$$
$$2\,dx - 3\,dy + 5\,dz = 0$$

Therefore $dx : dy : dz = 16 : 9 : -1$

235. Length of arc. 1. Let C denote a curve arc between $x = a$ and $x = b$, and let its equations be

$$y = f(x) \qquad\qquad z = \phi(x) \tag{1}$$

where $f'(x)$ and $\phi'(x)$ are supposed continuous.

As in § 57, inscribe in C a polygon of n sides all of which $\to 0$ when $n \to \infty$. We define the length of C as the limit approached by the perimeter of the polygon when $n \to \infty$. What follows shows that the limit exists.

Let PQ denote any side of the polygon, and Δx, Δy, Δz the projections of PQ on Ox, Oy, Oz. Then

$$PQ = [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]^{1/2}$$

By § 97 (6) there are two points, x_1 and x_2, in Δx such that

$$\Delta y = f'(x_1)\,\Delta x \qquad\qquad \Delta z = \phi'(x_2)\,\Delta x$$

Therefore $\quad PQ = [1 + f'^2(x_1) + \phi'^2(x_2)]^{1/2}\,\Delta x \qquad (2)$

Hence[1]

$$\begin{aligned}\lim \Sigma PQ &= \lim \Sigma[1 + f'^2(x_1) + \phi'^2(x_2)]^{1/2}\,\Delta x \\ &= \lim \Sigma[1 + f'^2(x_1) + \phi'^2(x_1)]^{1/2}\,\Delta x \\ &= \int_a^b [1 + f'^2(x) + \phi'^2(x)]^{1/2} dx\end{aligned}$$

Therefore the length s_a^b of the arc C is given by the formula:

$$s_a^b = \int_a^b [1 + f'^2(x) + \phi'^2(x)]^{1/2}\,dx \tag{3}$$

[1] For let M and m denote the greatest and least values of $\phi'^2(x)$ in Δx. Then

$$\phi'^2(x_2) = \phi'^2(x_1) + \theta(M - m) \qquad -1 < \theta < 1 \tag{1}$$

If A and B denote any two numbers such that $A > 1$ and $A + B > 0$, we have

$$|(A + B)^{1/2} - A^{1/2}| = |B \div [(A + B)^{1/2} + A^{1/2}]| < |B| \tag{2}$$

Hence, setting $A = 1 + f'^2(x_1) + \phi'^2(x_1)$ and $B = \theta(M - m)$, we have by (1)

$$|[1 + f'^2(x_1) + \phi'^2(x_2)]^{1/2} - [1 + f'^2(x_1) + \phi'^2(x_1)]^{1/2}| < |\theta| \cdot |(M - m)$$

Therefore

$$\lim \Sigma[1 + f'^2(x_1) + \phi'^2(x_2)]^{1/2}\Delta x = \lim \Sigma[1 + f'^2(x_1) + \phi'^2(x_1)]^{1/2}\Delta x,$$

since

$$\lim \Sigma\,|\theta|\,(M - m)\Delta x = 0.$$

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2. Replacing b by the x of any point P of C, we obtain

$$s = \int_a^x [1 + f'^2(x) + \phi'^2(x)]^{1/2}\, dx \tag{4}$$

which expresses the length s of the variable arc AP as a function of the x of its movable end point P. We have

$$\frac{ds}{dx} = [1 + f'^2(x) + \phi'^2(x)]^{1/2} \tag{5}$$

Hence ds/dx is positive at all points of C. This means that in giving the radical in (2) the + sign, we make the positive sense along the curve correspond to that along Ox.

3. We may regard the x, y, z of P as functions of the length s of the variable arc AP, and these functions have s-derivatives. For, by (4), (5), s is a one-valued, increasing, and differentiable function of x; hence, §§ 52, 54, x is a one-valued, increasing, and differentiable function of s. And, by hypothesis, y and z are differentiable functions of x, and therefore of s, § 40.

4. If we multiply (5) by dx and set $f'(x)\, dx = dy$, $\phi'(x)\, dx = dz$, we get

$$ds = [1 + f'^2(x) + \phi'^2(x)]^{1/2}\, dx = \pm\, [(dx)^2 + (dy)^2 + (dz)^2]^{1/2}$$

the $\pm$ sign being necessary because dx may be positive or negative. Hence, if the x, y, z of P are given as functions of a parameter t, of the kind described in § 234, and we take as the positive sense on the curve that which corresponds to increasing values of t, we have for the length s of the arc AP from $t = t_0$ to $t = t$:

$$s = \int_{t_0}^{t} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{1/2} dt \tag{6}$$

Since, by hypothesis, dx/dt, dy/dt, dz/dt do not vanish simultaneously, it follows from this formula, as from (4), that the x, y, z of P are differentiable functions of s.

Example 1. The length of $y = x$, $z = \frac{2}{3}x^{3/2}$ between $x = 0$ and $x = 2$ is $s = \int_0^2 (2 + x)^{1/2}\, dx = \frac{2}{3}(8 - 2\sqrt{2})$.

EXAMPLE 2. The length of the helix, $x = 3\cos\theta$, $y = 3\sin\theta$, $z = 4\,\theta$, between $\theta = 0$ and $\theta = \theta$ is $s = \int_0^\theta [9\sin^2\theta + 9\cos^2\theta + 16]^{1/2} d\theta$ $= 5\,\theta$. Hence also $\theta = \dfrac{s}{5}$, $x = 3\cos\dfrac{s}{5}$, $y = 3\sin\dfrac{s}{5}$, $z = \dfrac{4\,s}{5}$.

236. Theorem. *The limit of the ratio of an arc to its chord when the arc* $\rightarrow 0$, *is* 1.

For let Δs and PQ denote the lengths of the arc and chord. By § 235, 3., $\Delta x \rightarrow 0$ when $\Delta s \rightarrow 0$, and by § 235 (2), (4) and § 96 (7), there are points x_1, x_2, x_3 in Δx such that

$$PQ = [1 + f'^2(x_1) + \phi'^2(x_2)]^{1/2}\,\Delta x \qquad \Delta s = [1 + f'^2(x_3) + \phi'^2(x_3)]^{1/2}\,\Delta x$$

Hence $$\lim_{\Delta s \rightarrow 0} \frac{PQ}{\Delta s} = \lim_{\Delta x \rightarrow 0} \frac{[1 + f'(x_1)^2 + \phi'(x_2)^2]^{1/2}}{[1 + f'(x_3)^2 + \phi'^2(x_3)]^{1/2}} = 1$$

237. Direction cosines of the tangent. Let $\cos\alpha$, $\cos\beta$, $\cos\gamma$ denote the direction cosines of the tangent at P to the path of P, in the direction of the motion. Taking s as the independent variable, give the s of $P(x, y, z)$ the positive increment Δs, and let $Q(x + \Delta x,\ y + \Delta y,\ z + \Delta z)$ be the point corresponding to $s + \Delta s$. The direction cosines of PQ are $\Delta x/PQ$, $\Delta y/PQ$, $\Delta z/PQ$, and when $\Delta s \rightarrow 0$ they $\rightarrow \cos\alpha$, $\cos\beta$, $\cos\gamma$. Therefore, since $\Delta x/PQ = (\Delta x/\Delta s)(\Delta s/PQ)$ and so on, and $\Delta s/PQ \rightarrow 1$, we have

$$\cos\alpha = \frac{dx}{ds} \qquad \cos\beta = \frac{dy}{ds} \qquad \cos\gamma = \frac{dz}{ds} \tag{1}$$

We have $\dfrac{dx}{ds} = \dfrac{dx}{dt}\Big/\dfrac{ds}{dt} = \dfrac{dx}{dt}\Big/\left[\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2\right]^{1/2}$, and so on.

238. Directional derivatives. Let $u = f(x, y, z)$ be a given function of x, y, z. When, as in § 237, P moves along its path, s increasing, the value of u at P changes. The rate of change of u with respect to s at P is, § 237 (1),

$$\frac{du}{ds} = f_x\frac{dx}{ds} + f_y\frac{dy}{ds} + f_z\frac{dz}{ds} = f_x\cos\alpha + f_y\cos\beta + f_z\cos\gamma \tag{1}$$

Its value at any point P depends only on the values of f_x, f_y, f_z at P and the direction α, β, γ. It is therefore called the *directional derivative* of f in the direction α, β, γ.

EXAMPLE 1. The directional derivative of xyz in the direction for which $\cos\alpha = -1/3$, $\cos\beta = 2/3$, $\cos\gamma = -2/3$ is

$$(-yz + 2zx - 2xy)/3$$

EXAMPLE 2. The point P is moving on the curve $x = t$, $y = t^2$, $z = t^3$, t increasing. Show that when $t = -1$ the rate of change of $r = OP$ with respect to s is $-\sqrt{42}/7$.

EXAMPLE 3. The directional derivative of $f(x, y, z)$ at P in the direction of the normal to the surface $f(x, y, z) = c$ through P is called the *normal derivative* of f at P and is represented by df/dn. Show that $df/dn = \pm(f_x^2 + f_y^2 + f_z^2)^{1/2}$, and that it is greater numerically than any other directional derivative of f at P. The sign is $+$ for one direction on the normal, $-$ for the other. The vector whose x-, y-, z-components are f_x, f_y, f_z is called the *gradient* of f.

239. Velocity and acceleration. The discussion of curvilinear motion in §§ 82–87 is easily extended to space.

Let v_x, v_y, v_z denote the x-, y-, z-components of the velocity, v its magnitude, and α, β, γ its direction angles; and let a_x, a_y, a_z, a, and α', β', γ' have the corresponding meanings for the acceleration. Then

$$v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt} \qquad v = (v_x^2 + v_y^2 + v_z^2)^{1/2}$$

$$a_x = \frac{d^2x}{dt^2} \quad a_y = \frac{d^2y}{dt^2} \quad a_z = \frac{d^2z}{dt^2} \qquad a = (a_x^2 + a_y^2 + a_z^2)^{1/2}$$

$$\cos\alpha = \frac{v_x}{v}, \quad \cos\beta = \frac{v_y}{v}, \quad \cos\gamma = \frac{v_z}{v}$$

$$\cos\alpha' = \frac{a_x}{a}, \quad \cos\beta' = \frac{a_y}{a}, \quad \cos\gamma' = \frac{a_z}{a}$$

EXAMPLE 1. Discuss the motion of a point on the helix $x = 3\cos t$, $y = 3\sin t$, $z = 4t$, t denoting time.

$$v_x = -3\sin t, \quad v_y = 3\cos t, \quad v_z = 4 \qquad \therefore v = 5$$

and $$\cos\alpha = -\tfrac{3}{5}\sin t, \quad \cos\beta = \tfrac{3}{5}\cos t, \quad \cos\gamma = \tfrac{4}{5}$$

$$a_x = -3\cos t, \quad a_y = -3\sin t, \quad a_z = 0 \qquad \therefore a = 3$$

and $$\cos\alpha' = -\cos t, \quad \cos\beta' = -\sin t, \quad \cos\gamma' = 0$$

Hence the point moves with constant speed; the direction of the motion makes a constant angle with Oz; and the acceleration is of constant magnitude and is always directed toward Oz.

EXAMPLE 2. A point is moving on the curve whose equations are $xy + z = 0$, $2x + 3y + z = 5$. Find its velocity and acceleration vectors at $(2, 1, -2)$, if $v_x = 2$ and $a_x = 1$ at this point.

Differentiating both equations twice with respect to t, substituting the given values, and solving, we find

$$v_x, v_y, v_z = 2, -2, 2; \quad a_x, a_y, a_z = 1, -9, 25.$$

240. The osculating plane. Let C denote a given space curve, and P_1, P_2, P_3 three points on C. In general, when P_2, $P_3 \to P_1$ on C, the plane $P_1P_2P_3$ approaches a definite limiting position E. The plane E is called the *osculating plane* to C at P_1. Its equation may be found as follows:

Let the equations of C be

$$x = f(t) \qquad y = g(t) \qquad z = h(t) \tag{1}$$

Let the equation of the plane $P_1P_2P_3$ be

$$ax + by + cz + d = 0 \tag{2}$$

where a, b, c, d are undetermined constants, and consider the function

$$F(t) = a \cdot f(t) + b \cdot g(t) + c \cdot h(t) + d \tag{3}$$

Let P_1, P_2, P_3 correspond to $t = t_1, t_2, t_3$ in (1), and suppose that $t_1 < t_2 < t_3$. Then, P_1, P_2, P_3 being in the plane (2), $F(t)$ vanishes at $t = t_1, t_2, t_3$. Hence, by Rolle's theorem, § 96, $F'(t)$ vanishes at a point t'_1 between t_1 and t_2, and also at a point t'_2 between t_2 and t_3; and since $F'(t)$ vanishes at t'_1 and t'_2, $F''(t)$ vanishes at a point t''_1 between t'_1 and t'_2.

When $t_2, t_3 \to t_1$, then $t'_1, t'_2, t''_1 \to t_1$. Hence when $t_2, t_3 \to t_1$, and (2) becomes the equation of the osculating plane at P_1, the coefficients a, b, c, d satisfy the equations $F(t_1) = 0$, $F'(t_1) = 0$, $F''(t_1) = 0$, or setting $x_1 = f(t_1)$, $y_1 = g(t_1)$, $z_1 = h(t_1)$, the equations

$$ax_1 + by_1 + cz_1 + d = 0 \qquad a \cdot f'(t_1) + b \cdot g'(t_1) + c \cdot h'(t_1) = 0$$
$$a \cdot f''(t_1) + b \cdot g''(t_1) + c \cdot h''(t_1) = 0 \tag{4}$$

Hence the equation of the osculating plane may be found by eliminating a, b, c, d between (2) and (4). It is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ f'(t_1) & g'(t_1) & h'(t_1) \\ f''(t_1) & g''(t_1) & h''(t_1) \end{vmatrix} = 0 \tag{5}$$

When the equations of C are of the form $F_1(x, y, z) = 0$, $F_2(x, y, z) = 0$, we may take x as the parameter t. The second and third rows of (5) are then

$$1, \quad \frac{dy_1}{dx_1}, \quad \frac{dz_1}{dx_1}; \qquad 0, \quad \frac{d^2y_1}{dx_1^2}, \quad \frac{d^2z_1}{dx_1^2}$$

Evidently the tangent to the curve C at P_1 is in the osculating plane E. The normal to C at P_1 which lies in E is called the *principal normal*, and the normal to E at P_1 is called the *binormal* to C at P_1.

Thus, for the curve $y = x^2$, $z = x^3$ at O, Fig. 119, the plane $z = 0$ is the osculating plane E, Ox is the tangent, Oy the principal normal, and Oz the binormal. The curve passes through E at O, as was to be expected from the definition of E. Its projections on the xy-, xz-, and yz-planes are the curves $y = x^2$, $z = x^3$, and $z^2 = y^3$, the second of which has a point of inflection and the third a cusp at O.

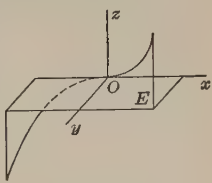

FIG. 119.

EXAMPLE. Find the equation of the osculating plane to the curve $x = t$, $y = t^2$, $z = t^3$ at the point $t = 1$; also the equations of the tangent, binormal, and principal normal at this point.

241. Curvature and torsion. In the equation § 240 (5), suppose the parameter t to be s, the length of the arc AP from some fixed point A on the curve C to the moving point P. Let ϕ and τ denote the angles which the tangent and the binormal to C at P make with some fixed direction. The s-rates of change of ϕ and τ, as s increases and P moves along C, are called the *curvature* and the *torsion* of C at P. Hence by definition

$$\text{Curvature} = \frac{d\phi}{ds} \qquad \text{Torsion} = \frac{d\tau}{ds} \tag{1}$$

The reciprocals of these numbers are called the *radius of curvature* and the *radius of torsion.*

Let l, m, n denote the direction cosines of the tangent at P. Then

$$\cos \Delta\phi = l(l + \Delta l) + m(m + \Delta m) + n(n + \Delta n)$$

But $l^2 + m^2 + n^2 = 1 \qquad (l + \Delta l)^2 + (m + \Delta m)^2 + (n + \Delta n)^2 = 1$

Hence $\quad (\Delta l)^2 + (\Delta m)^2 + (\Delta n)^2 = 2(1 - \cos \Delta\phi) = [2 \sin (\Delta\phi/2)]^2$

Therefore

$$\left(\frac{d\phi}{ds}\right)^2 = \lim_{\Delta s \to 0}\left[\frac{2 \sin (\Delta\phi/2)}{\Delta s}\right]^2 = \left(\frac{dl}{ds}\right)^2 + \left(\frac{dm}{ds}\right)^2 + \left(\frac{dn}{ds}\right)^2 \tag{2}$$

Similarly, if l_1, m_1, n_1 be the direction cosines of the binormal,

$$\left(\frac{d\tau}{ds}\right)^2 = \left(\frac{dl_1}{ds}\right)^2 + \left(\frac{dm_1}{ds}\right)^2 + \left(\frac{dn_1}{ds}\right)^2 \tag{3}$$

242. Indicatrix. Surface curvature. 1. Take any non-singular point of a surface S as the origin O, and the tangent plane to S at O as the xy-plane. For points of S in the neighborhood of O the expression for z in terms of x, y given by Taylor's theorem, § 228, 2, will be of the form

$$z = \tfrac{1}{2}(Ax^2 + 2\,Hxy + By^2) + R \tag{1}$$

in which we suppose that A, H, B are not all 0. It will be shown later, § 245, that, generally speaking, we can find a region about O in the xy-plane in which the value of R is as small as we please as compared with that of the bracketed term. And by a proper choice of the x-, y-axes we can give the bracketed expression the form $ax^2 + by^2$. Hence in the neighborhood of O, the surface S closely approximates a quadric surface

$$z = \tfrac{1}{2}(ax^2 + by^2) \tag{2}$$

and the sections of S by planes $z = k$ near the tangent plane $z = 0$ are similar to the conic

$$1 = \tfrac{1}{2}(ax^2 + by^2) \tag{3}$$

which is therefore called the *indicatrix* of S at O.

When a and b have the same sign, the indicatrix is an ellipse, real or imaginary, and, by (2), according as this sign is $+$ or $-$, S will lie wholly above or wholly below the plane $z = 0$ in the neighborhood of O. We then call O an *elliptic point*. Thus any point on a sphere.

When a and b have contrary signs, the indicatrix is a hyperbola whose asymptotes are the two lines represented by $ax^2 + by^2 = 0$, $z = 0$. It then follows from (2) that in one of the two pairs of opposite regions bounded by the planes $ax^2 + by^2 = 0$, S near O will lie above the plane $z = 0$, and in the other pair below it, and that S will cut the plane $z = 0$ in

curves which touch the lines $ax^2 + by^2 = 0$, $z = 0$ at O. We call O a *hyperbolic point.* Thus any point on a hyperbolic paraboloid (Fig. 112).

When b or a is 0, the indicatrix is a pair of parallel lines, and (2) represents a parabolic cylinder which touches the plane $z = 0$ along Ox or Oy. Hence O is called a *parabolic point* of S. But it will be shown later that when a or b is 0, the form of S near O may also depend on terms in the expansion of z which involve higher powers of x, y.

2. The curves in which S is cut by planes through the normal Oz are called the *normal sections* of S at O. It can be proved that one of the numbers a, b in (2) is the curvature of the section of greatest curvature at O and the other that of the section of least curvature.[1] The numbers $1/a$ and $1/b$ are called the *principal radii* of curvature of S at O. The product ab is called the *total curvature* of S at O. When ab is $+$ everywhere on S, S is called a *surface of positive curvature;* when $-$, a *surface of negative curvature.*

EXERCISE XLVI

1. Find the tangent planes and normals to the surfaces

$$(1)\ z = x^2 + xy - 2\,y^2 \text{ at } (1, 2, -5)$$
$$(2)\ x^2y + y^2z + z^2x = 1 \text{ at } (-1, 1, 0)$$

2. Find the points of the surface $x^2 + y^2 + 2\,yz + 2\,x = 0$ where the tangent plane is parallel to the xy-plane.

3. Find the equation of the cylindrical surface parallel to Oz and touching the surface $2\,xz + z^2 + y = 0$.

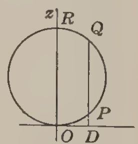

FIG. 120.

[1] Let the normal plane which makes the angle θ with Ox cut the xy-plane in OD, and S in the curve C. Let P be any point of C near O, and let OPR be the circle which touches C at O and passes through P. When $P \to O$ on C, OPR becomes the circle of curvature of C at O, § 97, Ex. 2. Therefore, since $OD^2 = DP \cdot DQ$, we have $\lim DP/OD^2 = 1/2\,\rho$, where ρ is the radius of curvature of C at O; and since $z = DP$, $x = OD \cos\theta$, $y = OD \sin\theta$, we have, by (2), $1/\rho = a\cos^2\theta + b\sin^2\theta$. This is Euler's formula for the curvature of any normal section. Evidently the greatest and least values of $1/\rho$ are a and b, or b and a.

4. Find the tangent line and normal plane to the curve $xy + yz = 4$, $x^2 + y^2 + z^2 = 14$ at $(-1, 2, 3)$.

5. At what angles does the curve $x = t$, $y = t^2$, $z = t^3$ meet the plane $3x - 7y + 2z = 0$?

6. Two surfaces $F_1 = 0$, $F_2 = 0$ are said to be *orthogonal* at a point of intersection P if their normals at P are perpendicular. Show that at P we then have

$$\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial z} = 0$$

7. What is the condition that the curve $F_1 = 0$, $F_2 = 0$ touch the surface $F_3 = 0$ at P?

8. Show that the condition that $F_1 = 0$, $F_2 = 0$, $F_3 = 0$ touch the same line at P is that F_1, F_2, F_3 and the functional determinant of F_1, F_2, F_3 with respect to x, y, z all be 0 at P.

9. Show that $x^2 + x - 2y + z = 0$, $y^2 - 2x + y + z = 0$, and $z^2 + x + y - 2z = 0$ touch the same line at O.

10. Find the length of the arc of $2y = x^2$, $3z = 4x^{3/2}$ between $x = 0$ and $x = 1$.

11. Show that as a moving point P passes through a point Q of a surface $F = 0$ where F_x, F_y, F_z are not all 0, the value of F at P changes sign.

12. Find the osculating plane to the curve $y = x^3$, $z = x^4$ at $(1, 1, 1)$; also the tangent, principal normal, and binormal.

13. Find the osculating plane to the curve $x^2 + y^2 + z^2 = 14$, $yz + zx + xy = 11$ at $(1, 2, 3)$.

14. Show that as P moves on the curve C, the acceleration vector PA lies always in the osculating plane at P; and that if the speed is constant, PA lies along the principal normal at P.

15. Prove the following relations between the s- and t-derivatives in curvilinear motion:

$$(1)\ \frac{ds}{dt} = \frac{dx}{ds}\frac{dx}{dt} + \frac{dy}{ds}\frac{dy}{dt} + \frac{dz}{ds}\frac{dz}{dt} \qquad (2)\ \frac{d^2x}{dt^2} = \frac{dx}{ds}\frac{d^2s}{dt^2} + \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2$$

16. If a particle of mass m be at O, and $r = OP$, then m/r is called the *potential* at P due to m. Prove that its directional derivative in any direction is the component in that direction of the force with which m attracts a unit particle at P.

17. Find the curvature and torsion at any point of the helix whose equations are $x = a \cos \theta$, $y = a \sin \theta$, $z = k\theta$.

18. Find the total curvature at O of the surface $z = 3x^2 - 4y^2$; of $z = x^2 - 4xy + 3y^2$.

19. If a plane through OD in Fig. 120, and making the angle ϕ with Oz, cuts S in C', the radius of curvature of C' at O equals that of the normal section C multiplied by $\cos\phi$. Prove this theorem.

ENVELOPES

243. Envelopes. In the xy-plane, an equation $f(x, y, a) = 0$ which involves an arbitrary constant or parameter a represents an infinite set or family of curves, there being one curve for each value of a. Let S denote this family of curves. It may be that a curve E exists which all the curves of S touch; we then call E the *envelope* of S.

The tangents to any given curve C constitute a family of lines of the type S, and the curve is its envelope. Let t_1 be a particular tangent, and t any neighboring tangent. When t rolls on C into coincidence with t_1, the point P where t cuts t_1 approaches a definite point A of t_1 as limit, namely the point where t_1 touches C. This fact suggests the following method of finding the envelope of any family of curves $f(x, y, a) = 0$, if it have one.

We assume that the values of x, y, a under consideration belong to a region R, § 214, 2. in which $f(x, y, a)$ and its partial derivatives of the first and second orders with respect to x, y, a are continuous.

Suppose that a representative curve (a) of S is met by a neighboring curve $(a + \Delta a)$ in a point P, and that when $\Delta a \to 0$, P approaches a definite non-singular point A of (a) as limit.

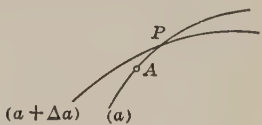

FIG. 121.

By the mean value theorem, § 97 (6), we have

$$f(x, y, a + \Delta a) - f(x, y, a) = f_a(x, y, a + \theta\,\Delta a)\,\Delta a \qquad (1)$$

Therefore, since the x, y of P satisfy $f(x, y, a) = 0$ and $f(x, y, a + \Delta a) = 0$, they also satisfy $f_a(x, y, a + \theta\,\Delta a) = 0$.

When $\Delta a \to 0$, then $f_a(x, y, a + \theta\,\Delta a) \to f_a(x, y, a)$, and $P \to A$. Hence the x, y of A satisfy the equations

$$f(x, y, a) = 0 \qquad (2) \quad \text{and} \quad f_a(x, y, a) = 0 \qquad (3)$$

and the locus L of all these limit points satisfies the equation got by eliminating a between (2) and (3).

We are to prove that, if $\partial f_a/\partial a \neq 0$ on L, then L touches each curve (a) of S at the limit point or points A on (a) and is therefore the envelope of S.

Unless $\partial f_a/\partial a$ is 0 at A, the equation $f_a(x, y, a) = 0$ has a solution for a in terms of x, y at A and in its neighborhood, § 226; let this solution be

$$a = \phi(x, y) \tag{4}$$

If we substitute (4) in $f(x, y, a) = 0$, we obtain an equation

$$f[x, y, \phi(x, y)] = 0 \tag{5}$$

which represents L at A and in its neighborhood.

Hence the slope of the tangent to L at A is that given by

$$f_x\,dx + f_y\,dy + f_a\,d\phi = 0 \tag{6}$$

But since f_a is 0 at A and $d\phi$ is finite, this equation reduces to

$$f_x\,dx + f_y\,dy = 0 \tag{7}$$

which also gives the slope of the curve (a) at A. Hence L touches (a) at A, as was to be proved.[1] Therefore

The envelope of S, if there be one, satisfies the equations $f = 0$, $f_a = 0$ and therefore also the equation $E(x, y) = 0$ got by eliminating a between $f = 0$ and $f_a = 0$.

The proof requires that the curves (a) and $(a + \Delta a)$ meet. But the intersections need not be real points. The proof also holds good when they are imaginary points.

[1] Observe that this conclusion would not follow were A a singular point of (a) since then both f_x and f_y would be 0 at A. It can be easily proved that if all the curves of S have singular points, the locus L' of these points also satisfies the equations $f = 0$ and $f_a = 0$; but ordinarily L' does not touch the curves of S.

One can also prove directly that if a curve C envelopes the curves $f(x, y, a) = 0$, then the x, y of any point P of C satisfy the equations $f = 0, f_a = 0$. For the x, y of P are functions of a, say $x = g(a), y = h(a)$, such that

$$f[g(a), h(a), a] \equiv 0 \quad (8) \qquad \text{and} \qquad h'(a)/g'(a) = -f_x/f_y \quad (9)$$

Differentiating (8) with respect to a and using (9) gives $f_a = 0$.

Example 1. Find the envelope of the family of circles

$$5x^2 + 5y^2 - 10ax + 4a^2 = 0 \quad (1)$$

Here $f_a = 0$ is

$$-10x + 8a = 0 \quad (2)$$

Eliminating a between (1), (2),

$$4y^2 - x^2 = 0$$

Hence the envelope is the pair of lines $y = x/2, y = -x/2$.

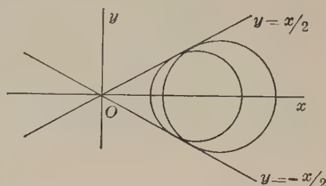

Fig. 122.

Example 2. Find the envelope of the circles through O with centers on $y^2 = x$.

A circle through O with center at (a, b) has the equation

$$x^2 + y^2 - 2ax - 2by = 0$$

But since (a, b) is on $y^2 = x$, we have $b^2 = a$. Hence

$$x^2 + y^2 - 2b^2x - 2by = 0$$

Eliminating b between $f = 0$ and $f_b = 0$ gives

$$y^2(2x + 1) + 2x^3 = 0$$

Example 3. Find the envelope of a line of constant length k whose ends move on Ox, Oy.

The intercept equation of the line is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad (1) \qquad \text{where } a^2 + b^2 = k^2 \quad (2)$$

Regarding b as a function of a, and differentiating (1), (2),

$$\frac{x}{a^2} + \frac{y}{b^2}\frac{db}{da} = 0, \; a + b\frac{db}{da} = 0 \quad \therefore \frac{x}{a^3} = \frac{y}{b^3} \quad \therefore \frac{x^{1/3}}{a} = \frac{y^{1/3}}{b} \quad (3)$$

The elimination of a, b between (1), (2), (3) gives $x^{2/3} + y^{2/3} = k^{2/3}$.

244. Envelopes of surfaces. 1. An equation of the form $f(x, y, z, a) = 0$ in which a is an arbitrary constant or parameter represents a family of surfaces S. Suppose that a representative surface (a) of S is cut by a neighboring surface

$(a + \Delta a)$ in a curve and that when $\Delta a \to 0$ this curve approaches a definite limiting position C. We then call C a *characteristic* of S. Let E denote the surface formed of all the characteristics of S. Under conditions similar to those stated in § 243 and by the reasoning of that section it can be proved that this surface E satisfies the equations

$$f = 0 \qquad \frac{\partial f}{\partial a} = 0 \tag{1}$$

and therefore the equation $E(x, y, z) = 0$ got by eliminating a between $f = 0$ and $f_a = 0$; also that E touches each surface (a) of S all along the characteristic C of (a). Hence E is called the *envelope* of S.

When f is of the first degree in x, y, z, and therefore S is a family of planes, the characteristics are straight lines and the envelope is called a *developable surface.* The osculating planes to a given curve in space form such a "one-parameter" family S; the envelope is the surface formed of the tangent lines to the given curve.

2. An equation $f(x, y, z, a, b) = 0$, in which a and b are arbitrary constants, represents a doubly infinite or *two-parameter family* of surfaces S. Under conditions analogous to those of § 243, it can be proved that a surface E exists, called the *envelope* of S, which touches each surface of S in one or more isolated points; and that E satisfies the equations

$$f = 0 \qquad \frac{\partial f}{\partial a} = 0 \qquad \frac{\partial f}{\partial b} = 0 \tag{2}$$

and therefore the equation $E(x, y, z) = 0$ got by eliminating a, b between these equations.

EXAMPLE 1. Show that the envelope of the one-parameter family of spheres $5x^2 + 5y^2 + 5z^2 - 10ax + 4a^2 = 0$ is the conical surface $4(y^2 + z^2) - x^2 = 0$.

EXAMPLE 2. Show that the envelope of the two-parameter family of spheres $(x - a)^2 + (y - b)^2 + z^2 = 2a + 2b$ is the surface

$$z^2 = 2(x + y + 1).$$

EXERCISE XLVII

1. Find the envelopes of the following families of curves:

1. $2x^2 + 2y^2 - 4\alpha x + \alpha^2 = 0$
2. $y^2 - \alpha x + \alpha^2 = 0$
3. $y = \alpha x + 1/\alpha$
4. $(x - \alpha)^2 + (y - \alpha)^2 = 2$
5. $(x - \alpha)^2 + (y - \alpha)^2 = 2\alpha$
6. $\alpha y^2 = x(x + \alpha)^2$

2. Circles are described on the double ordinates of the parabola $y^2 = 4x$ as diameters; show that the envelope of these circles is the parabola $y^2 = 4(x + 1)$.

3. The envelope of circles through O and with centers on $y = x^2 - x$ is $2y(x^2 + y^2) + (x - y)^2 = 0$.

4. If P is on $xy = c^2$, the envelope of circles with diameter OP is $(x^2 + y^2)^2 = 4c^2xy$.

5. The envelope of circles which touch Ox and have centers on $y = x^2$ is $y(2x^2 + 2y^2 - y) = 0$.

6. A straight line cuts Ox, Oy at A, B; find its envelope when it moves so that

1. OAB is constant
2. $OA + OB$ is constant.

7. Let $A(a, 0)$ and $B(0, b)$ be fixed points on Ox and Oy, P a variable point on the line segment AB, and PM, PN the perpendiculars from P on Ox, Oy. Show that the equation of the envelope of MN is

$$(x/a)^{1/2} + (y/b)^{1/2} = 1.$$

8. Let A be a fixed point on Ox, and P a variable point on Oy. Show that the envelope of the line through P perpendicular to AP is a parabola with vertex O and focus A.

9. The point P is on $y^2 = x$; PQ is perpendicular to OP; show that the envelope of PQ is $27y^2 = 4(x - 1)^3$.

10. Prove that the envelope of the ellipses $x^2/\alpha^2 + y^2/\beta^2 = 1$ for which $\alpha\beta = k$, is the pair of hyperbolas $4x^2y^2 = k^2$.

11. The locus of the centers of curvature of a curve $y = f(x)$ is called its *evolute*, § 81. The evolute is the envelope of the normals to the curve. Prove this by showing that if, taking the equation of the normal, $(dy_1/dx_1)(y - y_1) + (x - x_1) = 0$, we regard x_1, y_1 as parameters connected by $y_1 = f(x_1)$, and differentiate with respect to x_1, we are led to the parametric equations of the evolute, § 80 (4).

12. By the method of Ex. 11 prove that the evolute of $y = x^2$ is $27x^2 = 2(2y - 1)^3$.

13. The slope equation of the normal to the parabola $y^2 = 4ax$ is $y = m(x - 2a) - am^3$; find the evolute.

MAXIMA AND MINIMA

245. Maximum and minimum values. 1. A function $f(x, y)$ is said to have a *maximum value* at the point (a, b) if $f(a, b)$ is its greatest value in some region S, § 214, 6., about (a, b); that is, if

$$f(a+h, b+k)-f(a, b)<0 \tag{1}$$

for all values of h and k (except 0, 0) which are numerically less than some positive number d.

In like manner, $f(x, y)$ is said to have a *minimum value* at the point (a, b) if

$$f(a+h, b+k)-f(a, b)>0 \tag{2}$$

for all values of h, k such that $|h|, |k|<d$.

Similar definitions apply to functions of more than two variables.

2. *In a region S in which $f(x, y), f_x, f_y$ are continuous, $f(x, y)$ can be a maximum or minimum only at points where*

$$\frac{\partial f}{\partial x}=0 \quad \textit{and} \quad \frac{\partial f}{\partial y}=0 \tag{3}$$

For the statement that $f(x, y)$ has a maximum or minimum value at (a, b) implies that the x-function $f(x, b)$ has a maximum or minimum value at $x=a$, and therefore that $f_x(a, b)=0$, § 45. Similarly, $f_y(a, b)=0$.

A similar theorem holds good for $f(x, y, z)$, and so on.

3. At a point (a, b) where f_x, f_y are 0, $f(x, y)$ may be a maximum, or a minimum, or neither. We have the following test:

Suppose that $f(x, y)$ and its partial derivatives to those of the third order are continuous at (a, b) and in its neighborhood, that is, in some region S' about (a, b). Then, since $f_x(a, b)=0$ and $f_y(a, b)=0$, we have by Taylor's Theorem, § 222, if $(a+h, b+k)$ is in S',

$$f(a+h, b+k)-f(a, b)=\tfrac{1}{2}(Ah^2+2Bhk+Ck^2)+\tfrac{1}{6}R \tag{4}$$

where $A=\dfrac{\partial^2}{\partial a^2}f(a, b) \qquad B=\dfrac{\partial^2}{\partial a\,\partial b}f(a, b) \qquad C=\dfrac{\partial^2}{\partial b^2}f(a, b)$

and $R=\left(\dfrac{\partial}{\partial a}h+\dfrac{\partial}{\partial b}k\right)^3 f(a+\theta h, b+\theta k) \qquad 0<\theta<1$

The expression $Ah^2 + 2\,Bhk + Ck^2$ has a constant sign for all values of $h,k\ (\neq 0)$ if its factors are imaginary, that is, if $B^2 - AC < 0$ or $AC > B^2$ (which implies that A and C have the same sign); and this constant sign is that of A, or C. Moreover in this case we can find [1] a positive number d such that when $|h|, |k| < d$, then

$$\tfrac{1}{2}\,|Ah^2 + 2\,Bhk + Ck^2| > \tfrac{1}{6}\,|R| \tag{5}$$

and therefore

$$\text{sgn}\,[f(a+h, b+k) - f(a, b)] = \text{sgn}\,(Ah^2 + 2\,Bhk + Ck^2).$$ Hence

If $AC > B^2$, then $f(a, b)$ is a maximum or minimum: namely a maximum if $A < 0$, a minimum if $A > 0$.

If $AC < B^2$, then $Ah^2 + 2\,Bhk + Ck^2$ has real and distinct factors, and if $AC = B^2$, it has equal factors. In the first case, its sign is not constant and therefore $f(a, b)$ is not a maximum or minimum. In the second case, the sign of $f(a+h, b+k) - f(a, b)$ depends also on terms in higher powers of h, k.

The corresponding test for $f(x, y, z)$ at points where $f_x, f_y, f_z = 0$ is somewhat complicated and will not be given here.

EXAMPLE 1. Find the maxima and minima of $z = x^3 + y^2 - 3\,x$.

$$f_x = 3\,x^2 - 3, \quad f_y = 2\,y; \quad f_{xx} = 6\,x, \quad f_{xy} = 0, \quad f_{yy} = 2$$

The equations $f_x = 0, f_y = 0$ have the solutions $(1, 0)$ and $(-1, 0)$. At $(1, 0)$ we have $AC > B^2$ and $A > 0$; hence $f(1, 0) = -2$ is a minimum. At $(-1, 0)$ we have $AC < B^2$; hence $f(-1, 0) = 2$ is neither a maximum nor minimum.

The surface represented by $z = x^3 + y^2 - 3\,x$ touches the plane $z = -2$ at the point $(1, 0, -2)$ and lies above it in the neighborhood of the point. The surface also touches the plane $z = 2$ at the point $(-1, 0, 2)$ but lies partly above and partly below it in the neighborhood of the point. Compare § 242.

EXAMPLE 2. Test $z = x^2 + xy + y^2 + y^3$ for maxima and minima.

EXAMPLE 3. For $f(x, y) = (x - y)^2 - x^3$ show that f_x, f_y are 0 at $(0, 0)$, but that $f(0, 0)$ is not a maximum or minimum.

[1] For, setting $h = r\cos\phi$, $k = r\sin\phi$ gives

$$|Ah^2 + 2\,Bhk + Ck^2| = r^2\,|F(\phi)| \quad \text{and} \quad |R| = r^3\,|G(\phi)|$$

where $F(\phi)$ and $G(\phi)$ denote polynomials in $\sin\phi$, $\cos\phi$. Let m denote the least value of $|F(\phi)|$; and M the greatest value of $|G(\phi)|$ in S'. Then the inequality (5) will be satisfied when

$$\tfrac{1}{2}\,mr^2 > \tfrac{1}{6}\,Mr^3, \text{ that is, when } r < \frac{3\,m}{M}$$

4. In many cases it is unnecessary to use the test in 3. because of the following obvious theorem. (Compare § 47.)

Let S denote a given region in the xy-plane and C its boundary, and let $u = f(x, y)$, f_x, f_y be continuous in S. Then u has a greatest value in S, § 18, 1., and it takes this value either on C or at some interior point or points where $f_x, f_y = 0$. Hence

If there is one and but one point (a, b) within S at which $f_x, f_y = 0$, and if u has a greater value at (a, b) than anywhere on C, then $f(a, b)$ is the greatest value of u in S. Similarly for the least value of u in S.

There are corresponding theorems for $f(x, y, z)$, and so on.

EXAMPLE 4. The plane $x/a + y/b + z/c = 1$, where $a, b, c > 0$, cuts Ox, Oy, Oz at A, B, C. Find when the volume u of the rectangular parallelepiped which has three faces in the yz-, zx-, and xy-planes and a corner $P(x, y, z)$ in the triangle ABC, is greatest.

$$u = xyz \quad (1) \qquad \text{where} \qquad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad (2)$$

Because of (2), but two of x, y, z are independent; let these be x, y.

The x, y-function $u = cxy(1 - x/a - y/b)$, being continuous in the triangle OAB, has a greatest value in OAB. Furthermore u is positive within OAB and is 0 on the boundary. Hence if there be an interior point, and but one, where $\partial u/\partial x, \partial u/\partial y = 0$, then u takes its greatest value at that point.

Differentiating (1), (2) with respect to x and y, and setting $\partial u/\partial x$, $\partial u/\partial y = 0$,

$$yz + xy\frac{\partial z}{\partial x} = 0 \qquad \frac{1}{a} + \frac{1}{c}\frac{\partial z}{\partial x} = 0 \qquad (3)$$

$$xz + xy\frac{\partial z}{\partial y} = 0 \qquad \frac{1}{b} + \frac{1}{c}\frac{\partial z}{\partial y} = 0 \qquad (4)$$

Eliminating $\partial z/\partial x$ between the equations (3) gives $y(z/c - x/a) = 0$, and therefore, since y is $\neq 0$ within OAB, $z/c - x/a = 0$.

Similarly, it follows from the equations (4) that $z/c - y/b = 0$.

Hence $\quad \dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$ and therefore by (2) $\quad x = \dfrac{a}{3}, y = \dfrac{b}{3}, z = \dfrac{c}{3}$

Therefore the greatest value of u is $abc/27$.

EXAMPLE 5. Show that if $u = f(x, y, z)$ is a rational integral function, and when not 0 is positive, and if f_x, f_y, f_z are 0 at one and but one point (a, b, c), then $f(a, b, c)$ is the least value of u.

EXAMPLE 6. Find the point the sum of the squares of whose distances from three given points (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) is least.

246. Implicit functions. 1. In § 245, Ex. 4, an illustration is given of the method of finding the maximum or minimum values, if any, of a function of the type $u = f(x, y, z)$ when the variables x, y, z are connected by some relation $\phi(x, y, z) = 0$. The method may be extended to the case in which u is given as a function of n variables connected by $m(< n)$ equations.

EXAMPLE 1. Find the major and minor axes of the ellipse in which the ellipsoid $x^2 + 2y^2 + z^2 = 60$ is cut by the plane $x + y - z = 0$.

We seek the least and greatest values of

$$u = x^2 + y^2 + z^2 \qquad (1)$$

when x, y, z are connected by

$$x^2 + 2y^2 + z^2 = 60 \quad (2) \qquad \text{and} \qquad x + y - z = 0 \quad (3)$$

The equations (2), (3) define y and z as functions of x. Hence differentiating (1), (2), (3) with respect to x and setting $du/dx = 0$, we have

$$x + y\frac{dy}{dx} + z\frac{dz}{dx} = 0 \qquad x + 2y\frac{dy}{dx} + z\frac{dz}{dx} = 0 \qquad 1 + \frac{dy}{dx} - \frac{dz}{dx} = 0$$

Eliminating dy/dx and dz/dx, we get $y(x + z) = 0 \quad \therefore\ y = 0$ (4) or $x + z = 0$ (5)

Solving (2), (3), (4), we find $x^2 = 30, y^2 = 0, z^2 = 30 \quad \therefore$, by (1), $u = 60$.

Solving (2), (3), (5), we find $x^2 = 6, y^2 = 24, z^2 = 6 \quad \therefore$, by (1), $u = 36$.

Hence the lengths of the major and minor semiaxes are $2\sqrt{15}$ and 6. Their direction ratios are $1 : 0 : 1$ and $-1 : 2 : 1$.

2. The following example illustrates the procedure when the variable whose maximum or minimum is sought is involved implicitly in one or more of the given equations.

EXAMPLE 2. Find the points of the circle $x^2 + y^2 + z^2 = 6$ (1), $x + y + z = 0$ (2) which are at the greatest distances from the xy-plane.

We seek the points of the circle where $|z|$ is greatest. The equations (1) and (2) define any two of x, y, z as functions of the third. To solve our problem we must choose z as one of these dependent variables. Let y be the other, so that x is the independent variable. Then differentiating with respect to x, we have

$$x + y\frac{dy}{dx} + z\frac{dz}{dx} = 0 \qquad\qquad 1 + \frac{dy}{dx} + \frac{dz}{dx} = 0$$

Setting $dz/dx = 0$, and then eliminating dy/dx, we get $x - y = 0$ (3)

Solving the equations (1), (2), (3), we find the points $(-1, -1, 2)$ and $(1, 1, -2)$.

Hence the greatest distance of any point of the circle from the xy-plane is 2.

EXERCISE XLVIII

1. Find the maximum and minimum values of the following:

(1) $x^2 + 2xy + 2y^2 + 4x - 6y$ (2) $x^3 - y^3 + x^2 + y^2$

2. Show that the cube is the greatest rectangular parallelepiped of given surface area.

3. For what rectangular parallelepiped of given volume is the sum of the edges least?

4. Find the shortest distance between the lines $y = 2x$, $z = 3x$ and $y = x + 3$, $z = x$.

5. Find the rectangular parallelepiped of greatest volume that can be inscribed in the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

6. What is the greatest distance of a point of the curve $xyz = 1$, $x + 2y - z = 0$ from the xy-plane?

7. If $f(x, y, z) = 0$ be given, show that the maximum or minimum values of z, if any, are determined by the equations $f = 0, f_x = 0, f_y = 0$. What is the geometrical meaning of these conditions?

8. If $(x + y)^2 + (y + z)^2 + (z + x)^2 = 3$, show that the greatest and least values of z are $3/2$ and $-3/2$.

9. Divide a given positive number a into three parts x, y, z such that $x^m y^n z^p$ may be a maximum, m, n, p denoting positive constants.

10. A torpedo has the shape of a cylinder with conical ends. For given surface area what dimensions will give the greatest volume?

11. A house of given cubical content is to be constructed on a square base and with a gable roof. For what dimensions will the combined areas of the walls and roof be least?

12. Let ABC be a given triangle in the xy-plane. Respecting pyramids of given altitude h which have ABC for base, prove that the one of least surface area has its apex vertically above the center of the inscribed circle of ABC.

13. Let a, b, c denote the lengths of the sides of a given triangle, Δ its area, and x, y, z the perpendicular distances of an inner point from the sides. Show that the least value of $x^2 + y^2 + z^2$ is $4\,\Delta^2/(a^2 + b^2 + c^2)$.

14. The plane $x + y + z = 0$ cuts the hyperbolic paraboloid $z = xy$ in an hyperbola one branch of which passes through O, the other not. Find the point of this second branch which is nearest to O.

15. By the method of maxima and minima, find the point of the plane $ax + by + cz + d = 0$ which is nearest to O; also the point of the line of intersection of the planes $ax + by + cz + d = 0$, $a'x + b'y + c'z + d' = 0$ which is nearest to O.

16. The points P and P' are on two non-intersecting algebraic surfaces S and S'; show that when PP' is shortest it is normal to both S and S'.

17. Prove the corresponding theorem for two non-intersecting space curves C and C'.

18. Show that a polygon of n sides inscribed in a circle is greatest when it is regular.

19. Prove the following rule, called *Lagrange's method of multipliers*:

Let $u = f(x, y, z)$, where $\phi(x, y, z) = 0$. To find the values of x, y, z, if there be any, which make u a maximum or minimum, form the expression $F = f + \lambda\phi$, in which λ is an undetermined constant, and then solve the equations $F_x = 0$, $F_y = 0$, $F_z = 0$, $\phi = 0$ for x, y, z.

XXV. DIFFERENTIAL EQUATIONS

247. Differential equations. An equation which involves derivatives or differentials is called a *differential equation.*

Examples of such equations have been given in § 110. Other examples are :

1. $x\dfrac{dy}{dx} - y = 0$ or $x\,dy - y\,dx = 0$ 2. $z\,dx + x\,dy + y\,dz = 0$

3. $\dfrac{d^2y}{dx^2} + 4y = 0$ 4. $\left(x - y\dfrac{dy}{dx}\right)^2 = y$ 5. $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = z.$

If partial derivatives occur in the equation, it is called a *partial differential equation* (5.) ; if not, it is called an *ordinary differential equation* (1.–4.).

The order of the derivative of highest order in a differential equation is called the *order of the equation.* Thus 1., 2., 4., 5. are of the first order, and 3. is of the second order.

When an equation is rational and integral with respect to all the derivatives which occur in it, its degree with respect to the derivative of highest order is called the *degree of the equation.* Thus 1., 2., 3., 5. are of the first degree, and 4. is of the second degree.

248. Solutions of differential equations. 1. Let $F = 0$ denote a given differential equation in x, y, and x-derivatives of y. Any function y of x, whether expressed in the form $y = \phi(x)$ or the form $f(x, y) = 0$, which satisfies $F = 0$ identically, is called a *solution* of $F = 0$.

EXAMPLE 1. Prove that $y = e^{3x}$ is a solution of $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 0$.

If $y = e^{3x}$, then $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = (9 - 3 - 6)e^{3x} = 0$.

Example 2. Show that $y^2 - x + 1 = 0$ (1) is a solution of

$$4\,y\left(\frac{dy}{dx}\right)^2 - 2\,x\frac{dy}{dx} + y = 0 \quad (2)$$

Differentiating (1) gives $2\,y\dfrac{dy}{dx} - 1 = 0$ or $\dfrac{dy}{dx} = \dfrac{1}{2\,y}$ (3). Substituting (3) in (2) gives $\dfrac{1}{y} - \dfrac{x}{y} + y = \dfrac{y^2 - x + 1}{y}$, which $= 0$ because of (1).

2. From the equation $d^2y/dx^2 = 0$ (1) we obtain, by integration, $y = C_1x + C_2$ (2), where C_1, C_2 are arbitrary constants. This is the *general solution* of (1). Solutions, as $y = 3\,x$, $y = x - 5$, got by assigning particular values to the constants C_1, C_2 in (2) are called *particular solutions* of (1).

Similarly the general solution of $x\,dy/dx + y = 0$ is $xy = C$ (3).

We shall be chiefly concerned with differential equations $F = 0$ whose general solutions can, like (2) and (3), be obtained by the process of integration. We call them *integrable equations*. If such an equation is of the nth order, the number of integrations required to obtain its general solution will be n, and this solution will therefore involve n arbitrary constants.

3. If $f = 0$ denote an equation in x, y which involves n arbitrary constants algebraically, we can obtain the differential equation $F = 0$ of which it is the general solution by algebraically eliminating the n constants between $f = 0$ and the n equations got by differentiating $f = 0$ n times with respect to x.

Example 3. Find the differential equation whose general solution is $x^2 + y^2 + 2\,Cx = 0$.

Differentiating
$$x^2 + y^2 + 2\,Cx = 0 \tag{1}$$
gives
$$x + y\frac{dy}{dx} + C = 0 \tag{2}$$

Eliminating C between (1) and (2) gives

$$x^2 - y^2 + 2\,xy\frac{dy}{dx} = 0 \tag{3}$$

Geometrically both (1) and (3) represent the set of all circles which touch Oy at O

EXAMPLE 4. Find the differential equations of which the following are general solutions :

1. $y = C_1 + C_2x + C_3x^2$ 2. $y = C_1 \sin 2x + C_2 \cos 2x$

3. $Ax^2 + By^2 - 2x = 0$ 4. $(y - B)^2 = 4Ax$

4. To illustrate the sense in which a differential equation $F = 0$ in general, whether integrable or not, may be said to have solutions, consider the equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in a region S of the xy-plane in which f and its partial derivatives are real, one-valued, and continuous. Take any point $P(a, b)$ in S. The value of dy/dx at P is $f(a, b)$, and by successive differentiations of (1) we can find the values of d^2y/dx^2, d^3y/dx^3, $\cdots$ at P. Hence (1) and P determine a definite Taylor series :

$$y = b + \left(\frac{dy}{dx}\right)_{a,b} \frac{x-a}{1} + \left(\frac{d^2y}{dx^2}\right)_{a,b} \frac{(x-a)^2}{2!} + \cdots \tag{2}$$

It is proved later that for all values of x within a certain distance l of a, this series converges and satisfies (1) identically. Hence the function which it defines is a particular solution of (1). It may be called the solution of (1) at P.

If in (2) we replace b by an arbitrary constant C, we obtain an equation of the form $y = \phi(x, C)$ which represents the general solution of (1) within a certain rectangle R whose center is P and whose sides are parallel to Ox and Oy. Geometrically, it represents a set of curves, called *integral curves* of (1), one of which, and but one, passes through each inner point of R. A point Q moving across R in such a manner that the tangent of its direction angle always equals the value of $f(x, y)$ at Q will trace one of these curves.

For regions in which $\phi_C \neq 0$ we may regard $y = \phi(x, C)$ as expressible in the form $F(x, y) = C$, § 226.

EXAMPLE 5. Given the equation $\frac{dy}{dx} = x + y$. At the point $(0, C)$ we have

$$\frac{dy}{dx} = x + y = C, \qquad \frac{d^2y}{dx^2} = 1 + \frac{dy}{dx} = 1 + C,$$

$$\frac{d^3y}{dx^3} = \frac{d^2y}{dx^2} = 1 + C, \cdots, \qquad \frac{d^ny}{dx^n} = \frac{d^{n-1}y}{dx^{n-1}} = 1 + C.$$

Hence, by (2),

$$y = C + Cx + (C + 1)\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right]$$

This series converges for $|x| < \infty$, whatever the value of C may be. Hence it is the general solution of the given equation. It is equivalent to

$$y = (C + 1)e^x - (x + 1)$$

4. We may deal in a similar manner with an equation of the second order

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \tag{3}$$

Assign to x a particular value a, and arbitrary values C, C' to y, dy/dx, and then by aid of (3) compute d^2y/dx^2, d^3y/dx^3, $\cdots$ in terms of C, C' and form the corresponding Taylor series for y in powers of $x - a$. We thus obtain an equation of the type $y = \phi(x, C, C')$ which represents the general solution of (3) for values of x within a certain distance l of a, and values of C, C' between certain limits.

This discussion may be extended to equations of any order n and shows the sense in which it is true in general that the general solution of a differential equation of the nth order involves n arbitrary constants.

EXAMPLE 6. Find the solution of $\frac{d^2y}{dx^2} = y$ at $x, y, \frac{dy}{dx} = 0, C, C'$.

Here $y = \frac{d^2y}{dx^2} = \frac{d^4y}{dx^4} = \frac{d^6y}{dx^6} = \cdots = C, \quad \frac{dy}{dx} = \frac{d^3y}{dx^3} = \frac{d^5y}{dx^5} = \cdots = C'$

Hence $\quad y = C\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right] + C'\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]$

The series are convergent when $|x| < \infty$. Hence this is the general solution.

249. Graphical representation. The form and arrangement of the curves represented by a given differential equation $dy/dx = f(x, y)$ may sometimes be inferred quite readily from the equation itself.

EXAMPLE. Find the form of the integral curves of $\frac{dy}{dx} = y(1-x)$.

$$\frac{dy}{dx} = y(1-x) \quad (1) \qquad \therefore \frac{d^2y}{dx^2} = \frac{dy}{dx}(1-x) - y = yx(x-2) \quad (2)$$

Since $y = 0$ is a solution, Ox is one of the curves. Since (1) remains unchanged when y is replaced by $-y$, the family is symmetric to Ox.

For $y > 0$, we have

$$\frac{dy}{dx} \lesseqgtr 0 \text{ as } 1 - x \lesseqgtr 0 \ (3), \qquad \frac{d^2y}{dx^2} \lesseqgtr 0 \text{ as } x(x-2) \lesseqgtr 0 \ (4)$$

Hence above Ox the curves have maximum points on $x = 1(l)$, and points of inflection on $x = 0(l_1)$ and on $x = 2(l_2)$, and are convex downward between l_1 and l_2 and are elsewhere convex upward. No curve can meet Ox, since for points on Ox, $y = 0$ is the only solution; hence each curve tends to parallelism with Ox at the left and right. The curves below Ox are symmetric with those above Ox.

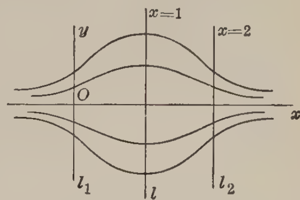

FIG. 123.

EXERCISE XLIX

1. Find the differential equations of the following families of curves:

1. All straight lines of slope ± 2. 2. All circles through O.

2. Find the equation of the family of curves defined by $\frac{d^2y}{dx^2} = 6x$; also of the set of these curves which pass through the point $(1, -1)$, and of the single curve which has the slope 3 at $(1, -1)$.

3. The slope of the radius vector from O to $P(x, y)$ is y/x. Hence show that

(1) $\frac{dy}{dx} = \frac{y}{x}$ represents the pencil of straight lines through O.

(2) $\frac{dy}{dx} = -\frac{x}{y}$ represents the family of circles whose center is at O.

(3) $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$ represents the family of circles through O and there tangent to Ox.

4. Express the particular solution of $dy/dx = x^2 + y^2$ whose graph passes through the point $(1, -1)$ by a Taylor series to the term involving $(x-1)^4$.

5. Sketch the family of curves represented by $dy/dx = x + y$, showing that the line $x + y + 1 = 0$ is one of them and that the others approach this line asymptotically.

EQUATIONS OF THE FIRST ORDER AND DEGREE

250. When the variables can be separated. An equation of the first order and degree $dy/dx = f(x, y)$ may be written in the differential form $M\,dx + N\,dy = 0$. It may happen that M and N are of the form $M = X_1Y_1$, $N = X_2Y_2$, where X_1, X_2 denote functions of x only, and Y_1, Y_2 functions of y only. It is then possible, by dividing by Y_1X_2, to reduce the equation to the form

$$\frac{X_1}{X_2}\,dx + \frac{Y_2}{Y_1}\,dy = 0$$

whose solution is
$$\int\frac{X_1}{X_2}\,dx + \int\frac{Y_2}{Y_1}\,dy = C \tag{1}$$

We call (1) the solution whether the integrations can be effected in finite terms or not.

EXAMPLE 1. Solve

$$\cos x \sin y\,dx + \sin x \cos y\,dy = 0$$

Dividing by $\sin y \sin x$, we get

$$\cot x\,dx + \cot y\,dy = 0$$

Integrating, $\log \sin x + \log \sin y = \text{const.}$

$$\therefore \log \sin x \sin y = \text{const.} \qquad \therefore \sin x \sin y = C.$$

EXAMPLE 2. Solve the following equations:

1. $x^2(y-1)\,dx + (xy+y)\,dy = 0$

2. $(y^2+1)\,dx - y(x^2+1)\,dy = 0$

3. $\dfrac{dy}{dx} = 2xy$

4. $(e^{x+y} - e^x)\,dx + (e^{x+y} + e^y)\,dy = 0$

5. $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

251. Homogeneous equations. When M and N are homogeneous functions of x and y, and of the same degree, $M\,dx + N\,dy = 0$ can be solved by aid of the substitution

$$y = vx \qquad dy = v\,dx + x\,dv$$

For in the resulting equation in x and v, M/N reduces to a function of v only, and the variables are therefore separable, as in § 250.

EXAMPLE 1. Solve $2\,xy\,dx + (y^2 - x^2)\,dy = 0$

Substituting $y = vx$, $2\,x^2v\,dx + (v^2x^2 - x^2)(v\,dx + x\,dv) = 0$

which reduces to $(v^3 + v)\,dx + (v^2 - 1)x\,dv = 0$

Separating variables and integrating, $\displaystyle\int \frac{dx}{x} + \int \frac{v^2 - 1}{v^3 + v}\,dv = \text{const.}$ (a)

By the method of partial fractions, § 118, $\displaystyle\frac{v^2 - 1}{v^3 + v} = -\frac{1}{v} + \frac{2\,v}{v^2 + 1}$ (b)

Hence (a) gives $\log x - \log v + \log (v^2 + 1) = \text{const.}$

Therefore $\dfrac{(v^2 + 1)x}{v} = C$, or setting $v = y/x$ and simplifying,

$$x^2 + y^2 = Cy$$

EXAMPLE 2. Solve the following equations:

1. $(2\,x + y)\,dx + (x + y)\,dy = 0$ **2.** $2\,xy\,dx + (y^2 - 2\,x^2)\,dy = 0$

3. $y^2\,dx = (xy - x^2)\,dy.$ **4.** $(\sqrt{xy} - x)\,dy + y\,dx = 0$

EXAMPLE 3. Solve $(x - y + 3)\,dx + (x + y + 1)\,dy = 0$ (a)

The lines $x - y + 3 = 0$ and $x + y + 1 = 0$ meet at the point $(-2, 1)$. The substitution $x = x' - 2$, $y = y' + 1$ in (a) gives the homogeneous equation

$$(x' - y')\,dx' + (x' + y')\,dy' = 0 \qquad (b)$$

Solving (b), as in Ex. 1, we get

$$\tan^{-1}\frac{y'}{x'} + \frac{1}{2}\log (x'^2 + y'^2) = C$$

Hence the solution of (a) is

$$\tan^{-1}\frac{y - 1}{x + 2} + \frac{1}{2}\log [(x + 2)^2 + (y - 1)^2] = C$$

All equations of the type $(ax + by + c)\,dx + (a'x + b'y + c')\,dy = 0$ can be solved by this method with the exception of those in which $a'x + b'y = k(ax + by)$, where k is a constant. In this exceptional case, the solution may be obtained by aid of the substitution

$$u = ax + by$$

EXAMPLE 4. Solve (1) $(2\,x + y - 4)\,dx + (x + y - 1)\,dy = 0.$

(2) $(2\,x + 3\,y)\,dx - (2\,x + 3\,y - 2)\,dy = 0.$

252. Linear equations. 1. A differential equation of the form

$$\frac{dy}{dx} + Py = Q \tag{1}$$

where P, Q are functions of x, is called a *linear equation* of the first order.

Let $\int P\,dx$ denote any particular integral of P. When the first member of (1) is multiplied by $e^{\int P\,dx}$, it becomes the derivative of $ye^{\int P\,dx}$; for

$$\frac{d}{dx} ye^{\int P\,dx} = \frac{dy}{dx} e^{\int P\,dx} + yPe^{\int P\,dx} = e^{\int P\,dx}\left[\frac{dy}{dx} + Py\right]$$

Hence we can find the solution of (1) by *multiplying both members by $e^{\int P dx}$ and then integrating with respect to x*, the result being

$$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + C \tag{2}$$

EXAMPLE 1. Solve $\dfrac{dy}{dx} + \dfrac{2}{x} y = x^3$.

$$\int P\,dx = \int \frac{2}{x}\,dx = 2\log x = \log x^2 \quad \therefore e^{\int P\,dx} = e^{\log x^2} = x^2.$$

Hence, by (2), we have

$$yx^2 = \int x^5\,dx + C = \frac{x^6}{6} + C, \quad \text{or} \quad y = \frac{x^4}{6} + \frac{C}{x^2}.$$

2. It follows from (1) and (2) that the solution of

$$\frac{d}{dx}\phi(y) + P\phi(y) = Q \text{ is } e^{\int P\,dx}\phi(y) = \int Qe^{\int P\,dx}\,dx + C \tag{3}$$

EXAMPLE 2. Solve $x \cos y \dfrac{dy}{dx} + (x + 1)\sin y = e^x$.

The equation may be written

$$\frac{d}{dx}\sin y + \left(1 + \frac{1}{x}\right)\sin y = \frac{e^x}{x}$$

$$\int P\,dx = \int\left(1 + \frac{1}{x}\right)dx = x + \log x \quad \therefore e^{\int P\,dx} = xe^x.$$

Therefore the solution is

$$xe^x \sin y = \int \frac{e^x}{x} \cdot xe^x\,dx + C, \quad \text{that is,} \quad xe^x \sin y = \frac{e^{2x}}{2} + C.$$

EXAMPLE 3. Show that the equation $\frac{dy}{dx} + Py = Qy^n$ may be reduced to the form $\frac{d}{dx} y^{-n+1} + (-n+1)Py^{-n+1} = (-n+1)Q$; and indicate its solution.

EXAMPLE 4. Show that the solution of $(x^2+1)\frac{dy}{dx} - xy = x^4y^3$ is

$$5(x^2+1) + (2x^5 + C)y^2 = 0.$$

EXAMPLE 5. Solve the following equations:

1. $\frac{dy}{dx} = x + y$
2. $\frac{dy}{dx} = x^2 + y$
3. $\frac{dx}{dy} = x + y$
4. $\sin x \frac{dy}{dx} + 2y\cos x = \sin 2x$
5. $x^2\frac{dy}{dx} + (1-2x)y = x^2$
6. $\frac{dy}{dx} + \frac{y}{\sqrt{x^2+1}} = \sqrt{x^2+1}$
7. $x\frac{dy}{dx} + y = y^2 \log^2 x$
8. $dy + (ax + by + c)\,dx = 0$
9. $\frac{dy}{dx} + \frac{y}{(x-1)(x-2)} = x^2 - 1$
10. $ydx + (2x - y^2)\,dy = 0$, which can be reduced to the form

$$\frac{dx}{dy} + P(y)\cdot x = Q(y)$$

253. Exact differential equations. When $M\,dx + N\,dy$ is the total differential of some function u of x and y, § 216, $M\,dx + N\,dy = 0$ is called an *exact differential equation.* It is then equivalent to $du = 0$ and has the solution $u = C$.

1. Evidently if $M\,dx + N\,dy$ can be reduced to a sum of known differentials, as

(1) $d(xy) = y\,dx + x\,dy$ (2) $d(x^2+y^2) = 2(x\,dx + y\,dy)$

$$(3)\quad d\frac{y}{x} = \frac{x\,dy - y\,dx}{x^2},$$

then $M\,dx + N\,dy = 0$ is exact, and its solution is obtainable immediately.

EXAMPLE 1. Solve $(2x - y)\,dx + (2y - x)\,dy = 0$.

Regrouping terms, $2(x\,dx + y\,dy) - (y\,dx + x\,dy) = 0$

$\therefore d(x^2+y^2) - d\,xy = 0 \quad \therefore x^2 + y^2 - xy = C.$

Example 2. Solve $(3x^2y - y)\,dx + (3x^3 + x)\,dy = 0$.

Regrouping terms, we obtain

$$3x^2(y\,dx + x\,dy) + (x\,dy - y\,dx) = 0$$

This is not an exact equation, but multiplying by $1/x^2$, we get

$$3(y\,dx + x\,dy) + \frac{x\,dy - y\,dx}{x^2} = 0$$

Hence $$3\,d(xy) + d\frac{y}{x} = 0 \quad \therefore 3xy + \frac{y}{x} = C$$

2. If, as in Ex. 2, $M\,dx + N\,dy = 0$ becomes an exact differential equation when multiplied by some function μ, then μ is called an *integrating factor* of $M\,dx + N\,dy = 0$.

Example 3. Show that $x\,dy - y\,dx = 0$ has the integrating factors $1/x^2$, $1/y^2$, $1/xy$, $1/(x^2 + y^2)$.

Example 4. Solve the following equations:

1. $x\,dy - y\,dx = (x^2 + y^2)x\,dx$ **2.** $y\,dx - x\,dy + y^2x\,dx = 0$

3. $(x\,dy + y\,dx)(y + 1) + x^2y^2\,dy = 0$ **4.** $x\dfrac{dy}{dx} = y + (x^2 + y^2)^{1/2}$

3. *The necessary and sufficient condition that $M\,dx + N\,dy = 0$ be an exact differential equation is that*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1}$$

For $M\,dx + N\,dy$ is a total differential when and only when a function u exists such that

$$\frac{\partial u}{\partial x} = M \quad (2) \qquad\qquad \frac{\partial u}{\partial y} = N \quad (3)$$

(1) It follows from (2) and (3), if M, N have continuous first partial derivatives, that

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial N}{\partial x}, \quad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence (1) is the necessary condition that $M\,dx + N\,dy = 0$ be an exact equation.

(2) But (1) is also the sufficient condition. The function $\int M\,dx$ got by integrating M with respect to x, regarding y as constant, satisfies (2). But

$$d\int M\,dx = M\,dx + \left[\frac{\partial}{\partial y}\int M\,dx\right]dy$$

$$\therefore M\,dx + N\,dy = d\int M\,dx + \left[N - \frac{\partial}{\partial y}\int M\,dx\right]dy$$

Hence $M\,dx + N\,dy$ is a total differential if $N - \dfrac{\partial}{\partial y}\displaystyle\int M\,dx$ is a function of y only, that is, if its partial derivative with respect to x is 0. But

$$\frac{\partial}{\partial x}\left[N - \frac{\partial}{\partial y}\int M\,dx\right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\int M\,dx = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

and this is 0 when $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence when $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then $M\,dx + N\,dy = 0$ is exact, and its solution is

$$\int M\,dx + \int\left[N - \frac{\partial}{\partial y}\int M\,dx\right]dy = C \qquad (4)$$

Example 5. Solve $(6\,x + 2\,y + 4)\,dx + (2\,x - 2\,y + 8)\,dy = 0$.

Here $\partial M/\partial y = 2$, $\partial N/\partial x = 2$; hence the equation is exact.

$$\int M\,dx = 3\,x^2 + 2\,xy + 4\,x, \qquad \frac{\partial}{\partial y}\int M\,dx = 2\,x$$

$$N - \frac{\partial}{\partial y}\int M\,dx = -\,2\,y + 8$$

Hence the solution is $3\,x^2 + 2\,xy + 4\,x - y^2 + 8\,y = C$.

Example 6. Show that the following are exact differential equations and solve them.

1. $(x + 2\,y)\,dx + (2\,x + 3\,y)\,dy$
2. $(ax + hy + g)\,dx + (hx + by + f)\,dy$
3. $(2\,xy - y^3)\,dx + (x^2 - 3\,xy^2)\,dy$
4. $2\,xy^3\,dx + \dfrac{3\,x^2y^3 + 1}{y}\,dy$

254. Applications. 1. The families of curves represented by the equations

$$F\left(x, y, \frac{dy}{dx}\right) = 0 \quad (1) \qquad\qquad F\left(x, y, -\frac{dx}{dy}\right) = 0 \quad (2)$$

are orthogonal. For the values of dy/dx given by (2) at any point $P(x, y)$ are the negative reciprocals of those given by (1); hence every curve of the family (2) cuts all curves of the family (1) at right angles, or is an *orthogonal trajectory* of the family (1); and vice versa. (See also § 110, Ex. 3.)

Example 1. Show that the orthogonal trajectories of the family $y^2 = Cx^3$ are the ellipses $x^2/3 + y^2/4 = C$.

2. The following examples illustrate the method of finding curves possessing assigned properties which can be expressed by differential equations.

EXAMPLE 2. Using polar coordinates, find the curves for which the angle ψ between the radius vector of any point and the tangent at the point is n times the vectorial angle θ.

By § 75 (1), $$r \Big/ \frac{dr}{d\theta} = \tan n\theta \quad \therefore \frac{dr}{r} = \cot n\theta \, d\theta \quad \therefore r^n = C \sin n\theta$$

EXAMPLE 3. Find the curves for which the y-intercept of the tangent at any point P equals OP.

From the equation § 34 (4) it follows that the y-intercept of the tangent at the point $P(x_1, y_1)$ is $y_1 - (dy_1/dx_1)\, x_1$. It may be $\pm$, but $OP = (x_1^2 + y_1^2)^{1/2}$ is always $+$. Therefore, omitting subscripts, we have
$$y - \frac{dy}{dx} x = \pm (x^2 + y^2)^{1/2}$$
whose solution is
$$x^2 + 2\,Cy - C^2 = 0 \tag{1}$$
Verify this result at the point (3, 4) for one of the curves (1) through this point.

EXAMPLE 4. Find the curves for which the length of the arc between $x = a$ and $x = x$ is k times the area bounded by the arc, the ordinates at $x = a$ and $x = x$, and Ox.

We have $$\int_a^x \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx = k \int_a^x y\, dx$$
or differentiating, $$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} = ky$$

By solving this equation algebraically for dy/dx and then separating the variables and integrating, we obtain
$$y = \frac{1}{2\,k}\left(Ce^{kx} + \frac{1}{C} e^{-kx}\right)$$
which represents a family of catenaries.

EXERCISE L

Solve the following equations:

1. $(y^3 + xy^3)\, dx + (x^3 - yx^3)\, dy = 0$
2. $2\,xy\, dx + (y^2 - 3\,x^2)\, dy = 0$
3. $\left(x - y \cos \frac{y}{x}\right) dx + x \cos \frac{y}{x}\, dy = 0$
4. $\frac{dy}{dx} + y \cos x = \frac{\sin 2\,x}{2}$
5. $3\,x^2 \frac{dx}{dy} - ax^3 = y + 1$

6. $(x + y)^2\, dy - dx = 0$ [Set $u = x + y$]

7. $x(x-1)\dfrac{dy}{dx} + (2x-1)y = x(x^2-1)$

8. $x\, dy + y\, dx = y^2 \log x\, dx$ 9. $(x^2y^2 + y)\, dx - x\, dy = 0$

10. $(x^2 + y^2)\, dx - x^2(x\, dy - y\, dx) = 0$

11. Find the orthogonal trajectories of the following families of curves

1. $x^2 + Cy^2 = 1$ 2. $x^2 + y^2 + 2Cx = 0$ 3. $y^m = Cx^n$

12. Find the curves which make the angle $\pi/4$ with the curves $xy = C$.

13. Find the curves for which the length of the normal equals that of its x-intercept.

14. Find the curves for which the subnormal equals the y-intercept of the tangent.

15. Let AB be a curve arc and $A'B'$ its projection on Ox, and let $ABB'A'$ be revolved about Ox. For what curve will the area generated by AB equal the volume generated by $ABB'A'$?

16. At the surface S of a certain convex lens whose axis is on Ox, rays parallel to Ox are refracted toward O. If the normal to S at P makes the angle α with Ox, and $\beta = x\,OP$, and the index of refraction is n, then $n = \sin\alpha/\sin(\alpha - \beta)$. Find the curve which will generate S when revolved about Ox.

17. Find the curve (a *tractrix*) whose tangent has the constant length l.

18. Show that if both $\phi(x, y) = C$, and $\psi(x, y) = C'$ are solutions of $M\, dx + N\, dy = 0$, then $N\phi_x - M\phi_y = 0$, $N\psi_x - M\psi_y = 0$; hence $\phi_x\psi_y - \phi_y\psi_x = 0$, and therefore ψ is a function of ϕ, § 232.

19. Show that if $F(x, y) = C$ is a solution of $M\, dx + N\, dy = 0$, then F_x/M equals F_y/N, and is an integrating factor of the equation.

20. As a hare running along Oy passes O, a hound, standing on Ox at the distance a to the right of O, starts in pursuit of him. The hound continually changes his direction so as always to face the hare, and runs twice as fast as the hare does. Show that if $p = dy/dx$ denote the slope at any point (x, y) of the hound's path C, then $\int_a^x (1 + p^2)^{1/2}\, dx = 2(y - px)$ and therefore, since also ds/dx is $-$ and y, $p = 0$ when $x = a$, the equation of C is

$$y = \frac{1}{3}\frac{x^{3/2}}{a^{1/2}} - a^{1/2}x^{1/2} + \frac{2}{3}a$$

21. A boat in being rowed across a river is kept directed toward a point on the shore directly opposite the starting point. If the rates of the current and the rowing are constant, a and b respectively, what is the path of the boat?

EQUATIONS OF THE FIRST ORDER OF HIGHER DEGREE

255. Geometrical interpretation. An equation of the form $f(x, y, C) = 0$, which is rational and integral and of the nth degree with respect to the arbitrary constant C, in general represents an infinite set of curves, n of which (real or imaginary) will pass through any assigned point (x_1, y_1).

If we eliminate C between the equations

$$f = 0 \text{ and } \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} p = 0, \text{ where } p = \frac{dy}{dx} \tag{1}$$

we get an equation of the form $\phi(x, y, p) = 0$, of the nth degree with respect to p, which represents the same set of curves.

Conversely, any differential equation $F(x, y, p) = 0$, of the first order and nth degree, represents an infinite set of curves, n of which (real or imaginary) will pass through any assigned point (x_1, y_1) and there have the n slopes got by solving $F(x_1, y_1, p) = 0$ algebraically for p. Generally speaking, this set of curves is not one that can be represented by an equation $f(x, y, C) = 0$ which is rational and integral with respect to C. But when it can be so represented, the equation $f(x, y, C) = 0$ is of the nth degree in C.

Example. The equation $(x - C)^2 + y^2 = 1$ represents the infinite set of circles of radius 1 with centers on Ox. Two of the circles pass through any point P whose ordinate is numerically less than 1.

Differentiating $\qquad (x - C)^2 + y^2 = 1$

gives $\qquad x + yp - C = 0$

Hence, eliminating C, $\qquad p^2y^2 + y^2 - 1 = 0$

a differential equation of the first order and second degree which also represents the set.

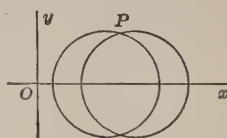

Fig. 124.

256. Equations solvable by first solving algebraically for p. It is sometimes possible to solve an equation $F(x, y, p) = 0$ algebraically for p in terms of x, y, and then to find the general

solutions of the resulting differential equations of the first degree $p = \phi_1(x, y)$, $p = \phi_2(x, y)$, $\cdots$. If these solutions are $f_1(x, y, C) = 0$, $f_2(x, y, C) = 0$, $\cdots$, the general solution of $F(x, y, p) = 0$ is

$$f_1(x, y, C) \cdot f_2(x, y, C) \cdots = 0$$

EXAMPLE 1. Solve the equation $xp^2 - (x^2 - y)p - xy = 0$.

Solving $xp^2 - (x^2 - y)p - xy = 0$ (1) for p gives $p = x$ (2) and $px + y = 0$ (3). The solutions of (2) and (3) are

$$y - \frac{x^2}{2} + C = 0 \text{ and } xy + C = 0$$

$$\therefore \text{ that of (1) is} \qquad \left(y - \frac{x^2}{2} + C\right)(xy + C) = 0$$

EXAMPLE 2. Solve $\quad yp^2 + 2\,xp - y = 0$

$$p = \frac{-x \pm (x^2 + y^2)^{1/2}}{y} \qquad \therefore \frac{y\,dy + x\,dx}{\pm (x^2 + y^2)^{1/2}} = dx$$

$$\therefore \pm (x^2 + y^2)^{1/2} = x + C$$

Hence the solution is $(x^2 + y^2) = (x + C)^2$, or $y^2 = 2\,Cx + C^2$.

EXAMPLE 3. Solve **1.** $(xp + y)^2 = x^3$ **2.** $xp^2 - 2\,yp - x = 0$ **3.** $(1 - x^2)p^2 = 4$

257. Equations solvable by first solving algebraically for y or x. 1. When the equation $F(x, y, p) = 0$ can be solved algebraically for y, it can be resolved into a set of one or more equations of the type

$$y = \psi(x, p) \tag{1}$$

It may be possible to solve (1) as follows: Differentiating with respect to x, we get

$$p = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial p}\frac{dp}{dx} \tag{2}$$

which does not involve y, and is an equation of the first order in p. Suppose that its solution is

$$\phi(x, p) = C \tag{3}$$

For any particular value of C, (1) and (3) may be interpreted as the parametric equations of a curve in terms of

the parameter p. The set of all such curves represents the solution of the equation (1).

To prove this, one must show for any curve represented by the parametric equations

$$y = \psi(x, t), \quad \phi(x, t) = C \tag{4}$$

that at the point $t = p$, its slope dy/dx is p. But from (4)

$$\frac{dy}{dt} = \frac{\partial\psi}{\partial x}\frac{dx}{dt} + \frac{\partial\psi}{\partial t} \qquad \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial t} = 0$$

Hence
$$\frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt} = \left[\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial t} - \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial t}\right]\bigg/\frac{\partial\phi}{\partial t} \tag{5}$$

On the other hand, substituting in (2) the value of dp/dx got by differentiating (3), we find

$$p = \left[\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial p} - \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial p}\right]\bigg/\frac{\partial\phi}{\partial p} \tag{6}$$

At the point $t = p$, the second member of (5) is identical with that of (6); hence at this point the slope dy/dx of the curve (4) equals p.

Example 1. Solve the equation $y = x + p^2$.

Differentiating with respect to x gives

$$p = 1 + 2\,p\frac{dp}{dx}$$

The general solution of this equation is

$$x = 2\,p + \log{(p-1)^2} + C$$

Hence the general solution of the given equation is

$$x = 2\,p + \log{(p-1)^2} + C \qquad y = 2\,p + p^2 + \log{(p-1)^2} + C$$

2. One may deal in a similar manner with an equation $F(x, y, p) = 0$ which can be solved algebraically for x. In differentiating we then replace dx/dy by $1/p$.

Example 2. Solve $\quad p^2 + py - x = 0$

Solving for x
$$x = py + p^2 \tag{a}$$

Differentiating with respect to y,

$$\frac{1}{p} = p + (2\,p + y)\frac{dp}{dy}$$

which reduces to the linear form

$$\frac{dy}{dp} - \frac{py}{1-p^2} = \frac{2\,p^2}{1-p^2}$$

Solving this equation,

$$y(1-p^2)^{1/2} = \sin^{-1}p - p\sqrt{1-p^2} + C \tag{b}$$

Hence (a) and (b) are the parametric equations of the required solution.

3. An equation of the type $y = px + f(p)$ is called a *Clairaut equation.* Its general solution is $y = Cx + f(C)$.

For differentiating $y = px + f(p)$ (1) with respect to x gives

$$\frac{dp}{dx}[x + f'(p)] = 0 \quad \therefore \frac{dp}{dx} = 0 \quad \text{or} \quad x + f'(p) = 0$$

The general solution of $dp/dx = 0$ is $p = C$; hence that of (1) is

$$y = Cx + f(C) \tag{2}$$

Since $x + f'(p) = 0$ does not involve dp/dx, it contributes nothing to the general solution of (1). Nevertheless if p be regarded as a parameter t, it can be shown by the reasoning in 1. that the curve

$$x = -f'(t) \qquad y = tx + f(t) \tag{3}$$

satisfies (1) regarded as a differential equation. The curve (3) is the envelope of the lines represented by the general solution (2), and (3) is called a singular solution of (1), § 258.

258. Singular solutions. 1. Let $F(x, y, p) = 0$ (1) denote a given equation of the second or a higher degree, and $f(x, y, C) = 0$ (2) its general solution. It may happen that the family of curves K represented by (1) or (2) is one which has an envelope, § 243, in other words, that a curve $\phi(x, y) = 0$ (3) exists which is touched at each of its points by one of the curves K. In that case, the slope at each point (x, y) of $\phi(x, y) = 0$ is the same as that of one of the curves K at the point, and therefore satisfies (1). Hence $\phi(x, y) = 0$ is a solution of (1). It is called a *singular solution,* because, generally speaking, it is not included among the solutions got by assigning particular values to C in the general solution

$$f(x, y, C) = 0.$$

EXAMPLE. The equation $p^2 - y = 0$ (1) has the general solution

$$4y = (x + C)^2 \tag{2}$$

The family of parabolas (2) has the envelope

$$y = 0 \tag{3}$$

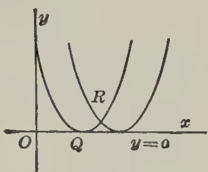

FIG. 125.

Hence $y = 0$ should be a singular solution of (1); and on testing it, we find that it is such a solution. For when $y = 0$, then $p = 0$; and the values $y = 0$, $p = 0$ satisfy (1). But $y = 0$ is not one of the curves (2).

2. If the family of curves K represented by $F(x, y, p) = 0$ has an envelope $\phi(x, y) = 0$, then at any point Q on $\phi(x, y) = 0$ two of the values of p given algebraically by $F(x, y, p) = 0$ are equal. For let (1) and (2) be curves of K. Their slopes at R are two of the values of p given by $F(x, y, p) = 0$ at R, and when $R \to Q$ along (1) and therefore (2) $\to$ (1), both slopes $\to$ the slope of (1) and $\phi(x, y) = 0$ at Q.[1]

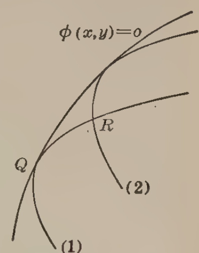

FIG. 126.

But when $p = p_1$ is a double root of $F = 0$ it is also a root of $\partial F/\partial p = 0$, § 50, 2. Therefore

All points (x, y) at which $F(x, y, p) = 0$ gives two equal values of p satisfy the equation $D(x, y) = 0$ got by eliminating p between

$$F(x, y, p) = 0 \textit{ and } \frac{\partial F}{\partial p} = 0$$

Hence the only possible singular solutions of $F(x, y, p) = 0$ are such as have equations of the form $E(x, y) = 0$ where E denotes D or some factor of D. If any such $E(x, y) = 0$ satisfies $F(x, y, p) = 0$, it is a singular solution. If no $E(x, y) = 0$ satisfies $F(x, y, p) = 0$, there is no singular solution. The first case is exceptional, the second general.

Ordinarily the family of curves K represented by the equation $F(x, y, p) = 0$ has no envelope, but the curves K have cusps, and $D(x, y) = 0$ represents the locus of these cusps.

For an equation of the second degree with respect to p, say $Ap^2 + Bp + C = 0$, we have $D(x, y) = B^2 - 4\,AC$.

EXAMPLE. Find the singular solution of $xp^2 - yp + a = 0$, if there be any.

$D = y^2 - 4\,ax = 0$ will be found to satisfy the equation; hence it is the singular solution. The general solution is $y = Cx + a/C$.

[1] This reasoning also holds good when the point of intersection of (2) with (1) is imaginary. See page 273.

EXERCISE LI

Solve the following equations, also finding singular solutions if any.

1. $y = p^2x + p^3$
2. $x = 2p + p^2$
3. $y - px + p^3 = 0$
4. $xp^2 - 2yp + x = 0$
5. $p^2 + px - y = 0$
6. $y + npx + ax^{n+1}p^n = 0$
7. $x = py + ap^2$
8. $yp^2 - 2xp + y = 0$
9. $y + 3xp + x^4p^3 = 0$
10. $y = 2px + ayp^2$ [set $z = y^2$]

11. In the case of $p^2 - x = 0$, show that $D = x = 0$ represents a locus of cusps.

12. The product of the perpendicular distances of the points $(-a, 0)$, $(a, 0)$ from the tangent to a curve C is a constant c. Show that the singular solution of $(y - px)^2 = c(1 + p^2) - a^2p^2$ is the equation of C and find it.

13. Find the curve for which the sum of the x- and y-intercepts of the tangent is constant.

LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

259. Linear equations. 1. A differential equation of any order n is said to be *linear* if it is of the first degree with respect to y and its derivatives. The general form of such an equation is :

$$\frac{d^ny}{dx^n} + X_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + X_{n-1}\frac{dy}{dx} + X_ny = X \tag{1}$$

where X_1, X_2, $\cdots$ X_n and X denote functions of x only, or constants.

Represent d/dx, d^2/dx^2, $\cdots$ by D, D^2, $\cdots$, and the expression $D^n + X_1D^{n-1} + \cdots + X_n$ by $f(D)$, and let $f(D)y$ mean the result of applying the operation represented by $f(D)$ to y; we may then write (1) in the form

$$(D^n + X_1D^{n-1} + \cdots + X_{n-1}D + X_n)y = X$$

or

$$f(D)y = X \tag{2}$$

2. Consider the *auxiliary equation* got by replacing X by 0 in (2), namely

$$(D^n + X_1D^{n-1} + \cdots X_{n-1}D + X_n)y = 0$$

or

$$f(D)y = 0 \tag{3}$$

If y_1, y_2 are functions of x, and c_1, c_2 are arbitrary constants, then $D^r(c_1y_1 + c_2y_2) = c_1D^ry_1 + c_2D^ry_2$, and therefore

$$f(D)[c_1y_1 + c_2y_2] = c_1f(D)y_1 + c_2f(D)y_2$$

The like is true for $c_1y_1 + c_2y_2 + c_3y_3$, and so on. Hence

If $y = y_1$, $y = y_2$, $\cdots$, $y = y_n$ are particular solutions of (3), *so that $f(D)y_1 \equiv 0$ and so on, and if c_1, c_2, $\cdots$, c_n are arbitrary constants, then*

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n \tag{4}$$

is also a solution of (3).

Suppose that y_1, y_2, $\cdots$, y_n are *linearly independent*, that is, that no particular set of constants c'_1, c'_2, $\cdots$, c'_n exists (except 0, 0, $\cdots$, 0) such that $c'_1y_1 + \cdots + c'_ny_n \equiv 0$. It is then not possible, by combining terms, to reduce the number n of arbitrary constants in (4), and (4) is the *general solution* of (3). For it can be proved that every particular solution of (3) can be got by assigning proper values to c_1, c_2, $\cdots$ c_n in (4). We call (4) the *complementary function* of the given equation (1).

3. *Again if y_1, y_2, $\cdots$, y_n have the same meanings as in* (4), *and if $y = Y$ denotes a particular solution of the given equation* (1), *the general solution of* (1) *is*

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + Y \tag{5}$$

For $f(D)[c_1y_1 + \cdots + c_ny_n + Y] = f(D)\,Y = X$; and every particular solution of (1) can be got by assigning proper values to c_1, c_2, $\cdots$, c_n.

Hence the problem of solving a linear equation $f(D)y = X$ reduces to that of finding n linearly independent particular

solutions of the auxiliary equation $f(D)y = 0$, and one particular solution of the equation $f(D)y = X$ itself. When the coefficients $X_1, X_2, \cdots, X_n$ are constants, $b_1, b_2, \cdots, b_n$, this can be done by the methods explained in the following sections.

260. The equation $f(D)y = 0$ with constant coefficients. 1. Suppose that

$$f(D) = D^n + b_1 D^{n-1} + \cdots + b_{n-1}D + b_n \tag{1}$$

where $b_1, b_2, \cdots b_n$ denote real constants; also that the roots of $f(D) = 0$, regarded as an algebraic equation in D, are $m_1, m_2, \cdots m_n$. We then have, algebraically,

$$f(D) \equiv (D - m_1)(D - m_2) \cdots (D - m_n) \tag{2}$$

But since $m_1, m_2, \cdots m_n$ are constants, it is also true, when D stands for d/dx, and y for a function of x, that

$$f(D)y \equiv (D - m_1)(D - m_2) \cdots (D - m_n)y \tag{3}$$

it being understood that the right member means that first the operation $D - m_n$ is to be applied to y, then $D - m_{n-1}$ to the result, and so on.

Thus, if $\quad f(D) = D^2 + b_1 D + b_2 = (D - m_1)(D - m_2)$

we have $\quad (D - m_1)(D - m_2)y = (D - m_1)(Dy - m_2 y)$

$$= D^2 y - (m_1 + m_2)Dy + m_1 m_2 y$$

that is, $\quad (D - m_1)(D - m_2)y = (D^2 + b_1 D + b_2)y$

The factors in (2), and therefore in (3), may be arranged in any order.

2. *The solution of each of the equations* $(D - m_1)y = 0, \cdots$ *is a solution of the equation* $f(D)y = 0$.

For let $D - a$ denote any one of the factors $D - m_1, \cdots$ in (3), and $\phi(D)$ the product of the remaining factors. Then if $y = y_1$ is a solution of $(D - a)y = 0$, we have

$$f(D)y_1 = \phi(D)(D - a)y_1 = \phi(D)0 = 0$$

By § 252, the solution of $(D - a)y = 0$ is $y = ce^{ax}$. Hence, § 259 (4),

When the roots $m_1, m_2, \cdots m_n$ are distinct, the general solution of $f(D)y = 0$ is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx} \tag{4}$$

EXAMPLE 1. Solve the equation $D^3y - D^2y - 2\,Dy = 0$.

The roots of $D^3 - D^2 - 2\,D = 0$ are $0, 2, -1$. Hence the general solution is $y = c_1 + c_2e^{2x} + c_3e^{-x}$.

3. Suppose that $r\ (> 1)$ of the roots equal a. The corresponding terms of (4) can then be combined in a single term ce^{ax} and therefore (4), since it involves less than n arbitrary constants, is not the general solution. But $(D - a)^r$ is a factor of $f(D)$, and it can be proved as in 2. that the solution of $(D - a)^ry = 0$ is a solution of $f(D)y = 0$.

The solution of $(D - a)^ry = 0$ is

$$y = e^{ax}(c_0 + c_1x + \cdots + c_{r-1}x^{r-1})$$

For $$(D - a)e^{ax}\phi(x) = e^{ax}\phi'(x)$$

Hence $$(D - a)^re^{ax}\phi(x) = e^{ax}\phi^{(r)}(x)$$

But when $\phi(x) = c_0 + c_1x + \cdots + c_{r-1}x^{r-1}$, then $\phi^{(r)}(x) = 0$. Therefore

When r of the roots equal a, the group of the corresponding terms of (4) is to be replaced by

$$e^{ax}(c_0 + c_1x + \cdots + c_{r-1}x^{r-1}) \tag{5}$$

EXAMPLE 2. Solve the equation $D^3y - 3\,Dy + 2\,y = 0$.

$D^3 - 3\,D + 2 = (D - 1)^2(D + 2)$. Hence $y = e^x(c_1 + c_2x) + c_3e^{-2x}$

4. The equation $f(D) = 0$ may have pairs of conjugate imaginary roots, as $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where $i = \sqrt{-1}$, § 19, 3. The corresponding terms $c_1e^{m_1x}$ and $c_2e^{m_2x}$ of (4) then have imaginary exponents. But from the definition of e^{x+iy} given in the chapter on functions of the complex variable it follows that $e^{(\alpha \pm i\beta)x} = e^{\alpha x} \cdot e^{\pm i\beta x}$ and that

$$e^{i\beta x} = \cos\beta x + i\sin\beta x \qquad e^{-i\beta x} = \cos\beta x - i\sin\beta x \tag{6}$$

and multiplying these equations by $c_1e^{\alpha x}$ and $c_2e^{\alpha x}$ and adding, we get

$$c_1e^{m_1x} + c_2e^{m_2x} = e^{\alpha x}(c'_1 \cos \beta x + c'_2 \sin \beta x) \qquad (7)$$

where $c'_1 = c_1 + c_2$ and $c'_2 = ic_1 - ic_2$.

When the roots m_1, m_2 occur r times, replace the constants c'_1, c'_2 in (7) by polynomials in x, like (5), of degree $r - 1$ and having arbitrary coefficients.

By adding and subtracting the formulas (6), we get

$$\cos \beta x = \frac{e^{i\beta x} + e^{-i\beta x}}{2} \qquad \sin \beta x = \frac{e^{i\beta x} - e^{-i\beta x}}{2\,i} \qquad (8)$$

EXAMPLE 3. Solve the equation $D^3y - 8\,y = 0$.

The roots of $D^3 - 8 = 0$ are $2,\ -1 \pm i\sqrt{3}$.

Hence $$y = c_1e^{2x} + e^{-x}(c_2 \cos \sqrt{3}\,x + c_3 \sin \sqrt{3}\,x)$$

EXAMPLE 4. Solve the equation $D^4y + 2\,D^2y + y = 0$.

The roots of $D^4 + 2\,D^2 + 1 = 0$ are $i,\ i,\ -i,\ -i$.

Hence $$y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x$$

EXAMPLE 5. Solve the following equations:

1. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 12\,y = 0$
2. $6\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2\,y = 0$
3. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} = 0$
4. $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} - 12\,y = 0$
5. $a\dfrac{d^2y}{dx^2} - (a^2+1)\dfrac{dy}{dx} + ay = 0$
6. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4\,y = 0$
7. $D^2(2\,D - 1)(D + 1)^3y = 0$
8. $(D^2 - 2\,D + 10)^2y = 0$

261. The equation $f(D)y = X$. Let $1/f(D)$ denote the *inverse* of the operation $f(D)$, that is, the operation which $f(D)$ undoes, so that

$$f(D)\left[\frac{1}{f(D)}X\right] = X \qquad (1)$$

Using this symbol $1/f(D)$, and § 260 (2), we may express the solution of the equation $f(D)y = X$ in the form

$$y = \frac{1}{f(D)}X = \frac{1}{D - m_1} \cdot \frac{1}{D - m_2} \cdots \frac{1}{D - m_n}X \qquad (2)$$

But if a denote any constant, we can find $[1/(D-a)]X$ as follows:

Set $\quad z = [1/(D-a)]X$. Then

$$(D-a)z = (D-a)\frac{1}{D-a}X, \text{ or } \frac{dz}{dx} - az = X$$

But this linear equation gives $z = e^{ax}\int e^{-ax}X\,dx$. Hence

$$\frac{1}{D-a}X = e^{ax}\int e^{-ax}X\,dx \tag{3}$$

The substitution of (3) in (2) for $a = m_n, m_{n-1}, \cdots m_1$ gives

$$y = e^{m_1x}\int e^{(-m_1+m_2)x}\int \cdots \int e^{(-m_{n-1}+m_n)x}\int e^{-m_nx}X\,(dx)^n \tag{4}$$

When constants of integration are introduced in performing the indicated integrations, (4) gives the general solution [1] of $f(D)y = X$. But in applying (4) we shall omit these constants. It then gives a particular solution, $y = Y$, of $f(D)y = X$. We obtain the general solution by adding Y to the complementary function obtained by the methods of § 260. See § 259 (5).

EXAMPLE 1. Solve $D^2y - 5\,Dy + 6\,y = e^x$.

Since $D^2 - 5\,D + 6 = (D-2)(D-3)$, we have

$$Y = e^{2x}\int e^{(-2+3)x}\int e^{-3x}\cdot e^x(dx)^2 = e^{2x}\int e^x\int e^{-2x}(dx)^2 = e^{2x}\int e^x\frac{e^{-2x}}{(-2)}\,dx = \frac{e^x}{2}$$

Hence the general solution is

$$y = c_1e^{2x} + c_2e^{3x} + \frac{e^x}{2}$$

EXAMPLE 2. Solve $(D-a)^ry = e^{ax}$.

$$Y = e^{ax}\int e^{(-a+a)x}\int \cdots \int e^{(-a+a)x}\int e^{-ax}\cdot e^{ax}(dx)^r = e^{ax}\int\int \cdots \int (dx)^r = e^{ax}\frac{x^r}{r!}$$

Hence the general solution is

$$y = e^{ax}\left(c_1 + c_2x + \cdots + c_{r-1}x^{r-1} + \frac{x^r}{r!}\right)$$

EXAMPLE 3. Solve the following equations:

1. $(D^2 - D - 6)y = e^{2x} + e^{4x}$
2. $(D-5)^3y = e^{5x}$
3. $(D^2 - 4)y = x$
4. $(D^3 - 5\,D^2 + 6\,D)y = e^x$

In the following cases there are simpler methods for finding the particular solution $[1/f(D)]X$.

[1] Since there are n integrations there are n constants and they enter linearly, as was stated in § 259.

262. When $X = e^{ax}$. Since $De^{ax} = ae^{ax}$, $D^2e^{ax} = a^2e^{ax}$, and so on, we have

$$f(D)e^{ax} = f(a)e^{ax}, \text{ and therefore } \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax} \tag{1}$$

the second formula being true since the results got by applying $f(D)$ to its two members are equal by the first formula.

When $f(a)$ is 0, (1) fails. But we then have $f(D) = (D - a)^r\phi(D)$ where $\phi(a) \neq 0$; hence, § 261, Ex. 2,

$$\frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)^r}\frac{1}{\phi(D)}e^{ax} = \frac{1}{(D-a)^r}\frac{e^{ax}}{\phi(a)} = \frac{e^{ax}}{\phi(a)}\frac{x^r}{r!}$$

EXAMPLE. Solve $\quad (D^2 - 1)^2y = e^{2x} + e^x.$

$$\frac{1}{(D^2-1)^2}e^{2x} = \frac{1}{(2^2-1)^2}e^{2x} = \frac{e^{2x}}{9}$$

$$\frac{1}{(D-1)^2}\frac{1}{(D+1)^2}e^x = \frac{1}{(D-1)^2}\frac{e^x}{4} = \frac{e^x}{4}\frac{x^2}{2}$$

Hence the general solution is

$$y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x} + \frac{e^{2x}}{9} + \frac{e^xx^2}{8}$$

263. When X is a polynomial in x of degree m. It is easily shown that in this case, when D is not a factor of $f(D)$, the Y given by § 261 (4) is of the form

$$A_0x^m + A_1x^{m-1} + \cdots + A_m \tag{a}$$

and when D^r is a factor of $f(D)$, of the form

$$x^r(A_0x^m + A_1x^{m-1} + \cdots + A_m) \tag{b}$$

Hence Y may be found by substituting (a) or (b) for y in $f(D)y = X$, and then determining the values of $A_0, A_1, \cdots, A_m$ which make the resulting equation an identity.

EXAMPLE 1. Solve $D^2y + y = x^3 + x^2$.

To find Y, substitute $y = Ax^3 + Bx^2 + Cx + E$, which gives

$$Ax^3 + Bx^2 + (C + 6A)x + (E + 2B) = x^3 + x^2$$

$$\therefore A = 1, B = 1, C = -6, E = -2$$

Hence $Y = x^3 + x^2 - 6x - 2$, and the general solution is

$$y = c_1 \cos x + c_2 \sin x + x^3 + x^2 - 6x - 2$$

EXAMPLE 2. Solve $D^2y - Dy = 8x^3$. To find Y, substitute

$$y = Ax^4 + Bx^3 + Cx^2 + Ex$$

EXAMPLE 3. Solve

1. $(D^3 - 8)y = x^2$ 2. $(D^2 - 2D + 1)y = x^2 + x$

264. When X is sin ax or cos ax. Observe that

$$D^2 \sin ax = -a^2 \sin ax \quad D^2 \cos ax = -a^2 \cos ax.$$

Hence in finding $[1/f(D)]X$, when X is sin ax or cos ax, we may replace D^2 by $-a^2$.

EXAMPLE 1. For the equation $(D^2 + 1)y = \sin 3x$, we have

$$Y = \frac{1}{D^2 + 1}\sin 3x = \frac{1}{-3^2 + 1}\sin 3x = -\frac{\sin 3x}{8}$$

EXAMPLE 2. Find Y for the equation $(D^2 + 2D + 5)y = \sin 2x$.

$$\frac{1}{D^2 + 2D + 5}\sin 2x = \frac{1}{2D + 1}\sin 2x = \frac{2D - 1}{4D^2 - 1}\sin 2x$$

$$= \frac{2D - 1}{-17}\sin 2x = -\frac{1}{17}(4\cos 2x - \sin 2x)$$

EXAMPLE 3. Solve

1. $(D^2 - 5D + 6)y = \sin 3x + \cos 4x$ 2. $(D^4 - D^2)y = \sin x$

This method fails when $D^2 + a^2$ is a factor of $f(D)$. We may then express sin ax or cos ax in terms of e^{iax} and e^{-iax}, § 260 (8), and then proceed as in § 262.

265. When X is $e^{ax}V(x)$. Since $D = a + (D - a)$ we can express $f(D)$ in the form $F(D - a)$, § 186. By § 260, 3., we have $(D - a)^r e^{ax}V = e^{ax}D^rV$, and therefore

$$F(D - a)e^{ax}V = e^{ax}F(D)V$$

But $F(D - a) = f(D)$, and therefore $F(D) = f(D + a)$. Hence

$$f(D)e^{ax}V = e^{ax}f(D + a)V$$

and therefore
$$\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D + a)}V \tag{1}$$

the second formula being true because the results got by applying $f(D)$ to its two members are equal by the first.

EXAMPLE 1. Find a particular solution of $(D^2 + 1)y = e^x \sin 2x$.

$$Y = \frac{1}{D^2+1} e^x \sin 2x = e^x \frac{1}{(D+1)^2+1} \sin 2x$$

$$= e^x \frac{1}{D^2 + 2D + 2} \sin 2x = -\frac{e^x}{10}(2\cos 2x + \sin 2x)$$

EXAMPLE 2. Solve

1. $D^2y + 5Dy + 4y = e^{-x}x^2$ 2. $D^3y = e^x(\sin x + \cos x)$

266. Variation of parameters. The following example illustrates a method of deriving the general solution from the complementary function. It is applicable to linear equations with constant or variable coefficients.

EXAMPLE 1. The complementary function of

$$D^2y + y = \tan x \quad (1) \quad \text{is} \quad y = C_1 \sin x + C_2 \cos x \quad (2)$$

Let us inquire whether if C_1, C_2 be regarded as functions of x instead of constants, they can be so determined that (2) will represent the general solution of (1).

Differentiating (2),

$$Dy = C_1 \cos x - C_2 \sin x + \frac{dC_1}{dx}\sin x + \frac{dC_2}{dx}\cos x$$

Suppose C_1, C_2 to be such functions that

$$\frac{dC_1}{dx}\sin x + \frac{dC_2}{dx}\cos x = 0 \quad (3)$$

so that (as when C_1, C_2 are constants)

$$Dy = C_1 \cos x - C_2 \sin x \quad (4)$$

Differentiating (4) gives

$$D^2y = -C_1 \sin x - C_2 \cos x + \frac{dC_1}{dx}\cos x - \frac{dC_2}{dx}\sin x \quad (5)$$

Substituting (5) and (2) in (1), we obtain

$$\frac{dC_1}{dx}\cos x - \frac{dC_2}{dx}\sin x = \tan x \quad (6)$$

Hence (2) will satisfy (1) if C_1 and C_2 satisfy (3) and (6).

Solving (3), (6) algebraically for $\frac{dC_1}{dx}$, $\frac{dC_2}{dx}$, we get

$$\frac{dC_1}{dx} = \sin x \qquad \frac{dC_2}{dx} = -\sin x \tan x \quad (7)$$

Integrating the equations (7) gives

$$C_1 = -\cos x + c_1 \qquad C_2 = -\log(\sec x + \tan x) + \sin x + c_2 \qquad (8)$$

where c_1, c_2 are constants. And substituting (8) in (2), we get the general solution of (1)

$$y = c_1 \sin x + c_2 \cos x - \cos x \log(\sec x + \tan x) \qquad (9)$$

EXAMPLE 2. The complementary function of

$$x^2 D^2y - 2x\,Dy + 2y = x^3 \quad \text{is} \quad y = C_1x + C_2x^2$$

By the method of variation of parameters show that the general solution is $y = c_1x + c_2x^2 + x^3/2$.

An equation like this, in which $f(D)$ is a sum of terms of the type x^rD^r, is sometimes called a *homogeneous equation.* It can be reduced to a linear equation with constant coefficients by the substitution $x = e^t$.

EXERCISE LII

Solve the following equations:

1. $D^3(2D-5)(D^2+D+1)^2y=0$
2. $(2D^2 - D - 6)y = e^{3x}$
3. $(D-2)^2y = 2e^x + e^{2x}$
4. $(D^2 + D)y = e^x + 3$
5. $(D^2 + 7D + 12)y = 6x + 10$
6. $(D^2 + 2D + 2)y = x^2$
7. $(D^2 + 2D)y = x^2$
8. $(D + 4)y = 8x^3 + 2x^2 + 5$
9. $(D^2 + D)y = \sin 2x + \cos x$
10. $(D^3 + D^2 - D - 1)y = \sin x$
11. $(D^2 + 1)y = \sin x$
12. $(D^2 + D - 6)y = e^xx^2$
13. $D^2y = e^{3x}\cos 2x$
14. $(D^2 + 1)y = \sec x$ (by § 266)
15. Find the integral curve of $D^2y - 2Dy + 2y = \cos x$ which touches Ox at O.

EQUATIONS OF THE SECOND ORDER

267. Equations of the second order. 1. Any equation of the second order in y which does not involve y explicitly may be reduced to an equation of the first order in p by the substitution

$$\frac{dy}{dx} = p \qquad \frac{d^2y}{dx^2} = \frac{dp}{dx} \qquad (1)$$

2. An equation of the second order in y which does not involve x explicitly may be reduced to one of the first order in p by the substitution

$$\frac{dy}{dx} = p \qquad \frac{d^2y}{dx^2} = \frac{dp}{dy}p \tag{2}$$

For
$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}p$$

EXAMPLE 1. Substituting (2) in $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 1$

gives
$$yp\frac{dp}{dy} + p^2 = 1$$

Hence $(1 - p^2)y^2 = c_1 \quad \therefore \dfrac{dy}{dx} = \dfrac{(y^2 - c_1)^{1/2}}{y} \quad \therefore y^2 - (x + c_2)^2 = c_1$

3. Let $f(D)y = X$ be any linear equation of the second order, and let $y = y_1$ be a solution of $f(D)y = 0$. It will be found that the substitution $y = y_1v$ reduces $f(D)y = X$ to a linear equation in v which lacks the v term and is therefore of the type 1.

EXAMPLE 2. Solve $(x^2 + 1)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = (x^2 + 1)^2$.

The auxiliary equation has the solution $y = x$.
Substituting $y = vx$ gives

$$\frac{d^2v}{dx^2} + \frac{2}{x(x^2+1)}\frac{dv}{dx} = \frac{x^2+1}{x}$$

whose solution is
$$v = \frac{x^3}{6} + \frac{x}{2} + c_1\left(x - \frac{1}{x}\right) + c_2$$

Hence,
$$y = \frac{x^4}{6} + \frac{x^2}{2} + c_1(x^2 - 1) + c_2x$$

268. Equations of motion. A particle P of mass m is moving in a straight line under the action of forces the sum of whose components along the line is F. Let α denote the acceleration of the motion, and take as the unit of force the force which gives unit mass unit acceleration. Then, if F

denote the measure of the force F in terms of this unit, we have by Newton's laws of motion

$$F = m\alpha \tag{1}$$

Taking the line in which P moves as x-axis, (1) may be written

$$m\frac{d^2x}{dt^2} = F \qquad \text{or} \qquad mv\frac{dv}{dx} = F \tag{2}$$

Suppose F to be given as a constant or a known function of t, x, v. The differential equation (2), and any one set of corresponding values of t, x, v will then completely determine the motion of P.

In like manner, if P move in space under the action of forces the sums of whose x-, y-, z-components are X, Y, Z, then

$$m\frac{d^2x}{dt^2} = X \qquad m\frac{d^2y}{dt^2} = Y \qquad m\frac{d^2z}{dt^2} = Z \tag{3}$$

These equations and the position and velocity of P at any given instant $t = t_0$ completely determine the motion of P.

Example. A body of mass m is projected vertically upward with the initial velocity v_0. Discuss the motion on the supposition that the resistance of the air is proportional to the speed.

Take the vertical as x-axis, the point of departure as the origin, and the downward direction as positive. We then have

$$m\frac{d^2x}{dt^2} = mg - kv \tag{1}$$

where g is the acceleration of gravity and k is some positive constant.

Hence dividing by m, and setting $k/m = \mu$,

$$\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} = g \tag{2}$$

The complementary function of (2) is $x = c_1 + c_2e^{-\mu t}$; and $x = (g/\mu)t$ is a particular integral. Hence the complete solution is

$$x = c_1 + c_2e^{-\mu t} + \frac{g}{\mu}t \tag{3}$$

and this gives

$$v = -c_2\mu e^{-\mu t} + \frac{g}{\mu} \tag{4}$$

When $t = 0$, then $x = 0$ and $v = v_0$

hence
$$0 = c_1 + c_2 \quad \text{and} \quad v_0 = -c_2\mu + \frac{g}{\mu}$$

Solving these equations for c_1, c_2 and substituting in (3), we get finally

$$x = \left(\frac{v_0}{\mu} - \frac{g}{\mu^2}\right) + \frac{g}{\mu}t + \left(\frac{g}{\mu^2} - \frac{v_0}{\mu}\right)e^{-\mu t} \tag{5}$$

which gives the height x at any instant t.

269. Simple harmonic motion. This name is given to the motion of a particle P on a line under the action of a force which is directed toward a fixed point O on the line and is proportional to the distance of P from O. Taking the line as x-axis and O as origin, we can reduce the equation of motion to the form $d^2x/dt^2 = -k^2x$, or

$$\frac{d^2x}{dt^2} + k^2x = 0 \tag{1}$$

By § 260 (7), the general solution of this equation is

$$x = c_1 \cos kt + c_2 \sin kt \tag{2}$$

or
$$x = a \cos (kt - \beta) \tag{3}$$

where

$$a = (c_1^2 + c_2^2)^{1/2} \quad \cos\beta = \frac{c_1}{(c_1^2 + c_2^2)^{1/2}} \quad \sin\beta = \frac{c_2}{(c_1^2 + c_2^2)^{1/2}}$$

If the position and velocity of P at any instant $t = t_0$ be given, we can find c_1, c_2 or a, β from (2) or (3) and the corresponding equation for v as in the example in § 268.

As (3) shows, the motion is *oscillatory*. As t increases from the value β/k, P moves backward from $x = a$ to $x = -a$, then forward to $x = a$, and so on. The time or *period* of each complete oscillation is $2\pi/k$. The distance a is called the *amplitude*.

EXAMPLE 1. When (3) is $x = 3 \cos 2t$, we have $v = -6 \sin 2t$. Find x and v for each of the following values of t: $0, \pi/4, \pi/2, 3\pi/4, \pi$.

EXAMPLE 2. If $d^2x/dt^2 + 4x = 0$, and $x = 3$, $v = 8$ when $t = 0$, show that
$$x = 3 \cos 2t + 4 \sin 2t = 5 \cos (2t - .94)$$

270. Damped harmonic motion. Suppose that a body in simple harmonic motion is subjected to a resistance which is proportional to the speed. Its equation of motion can then be reduced to the form

$$\frac{d^2x}{dt^2} + 2\,\mu\frac{dx}{dt} + k^2x = 0 \tag{1}$$

where μ denotes a positive constant. The roots of

$$D^2 + 2\,\mu D + k^2 = 0 \qquad \text{are} \qquad -\,\mu \pm \sqrt{\mu^2 - k^2}$$

Suppose $\mu^2 < k^2$ and set $k^2 - \mu^2 = \gamma^2$. Then, § 260 (7), the general solution of the equation (1) is

$$x = e^{-\mu t}[c_1 \cos\gamma t + c_2 \sin\gamma t] = Ce^{-\mu t}(\cos\gamma t - \delta) \tag{2}$$

where C and δ denote constants whose values can be found when any set of corresponding values of t, x, v is given.

Hence the motion continues to be oscillatory. But since $\gamma < k$, the period is increased; and since $e^{-\mu t}$ decreases as t increases, the amplitude continually decreases.

271. Forced vibrations. Suppose that a body in simple harmonic motion is subjected to a disturbing force which is itself periodic, being such that the reduced equation of motion becomes

$$\frac{d^2x}{dt^2} + k^2x = h\cos\gamma t \tag{1}$$

where h and γ are known constants. By §§ 264, 269, the solution of (1) is

$$x = a\cos(kt - \beta) + \frac{h}{k^2 - \gamma^2}\cos\gamma t \tag{2}$$

The motion is compounded of two simple harmonic motions. The case of most interest is that in which k and γ are nearly equal. The amplitude $h/(k^2 - \gamma^2)$ is then very large and the oscillations or vibrations of the body are therefore violent. It is for this reason that soldiers break step when crossing a bridge, lest their steps be in tune with the natural oscillations of the bridge.

272. Curvilinear motion about a center of force. A particle P moving about O under the action of a force directed toward O and inversely proportional to OP^2 describes a conic of which O is a focus.

For take any rectangular axes Ox, Oy. Let r denote the length of the vector OP and θ its direction angle referred to Ox. Also let PT denote a vector perpendicular to OP and having the direction angle

$\theta + \pi/2$. Finally let A_x, A_y, A_{OP}, A_{PT} denote the components of the acceleration in the directions Ox, Oy, OP, PT respectively. Then

$$A_{OP} = A_x \cos\theta + A_y \sin\theta \qquad A_{PT} = -A_x \sin\theta + A_y \cos\theta$$

which, when A_x, A_y are expressed in terms of r, θ, t by aid of the relations $x = r\cos\theta$, $y = r\sin\theta$, reduce to the form

$$A_{OP} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \qquad A_{PT} = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

By hypothesis, $A_{OP} = -\mu/r^2$, where μ denotes some positive constant, and $A_{PT} = 0$. Hence the equations of motion of P are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2} \quad (1) \quad \text{and} \quad 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0 \qquad (2)$$

The solution of (2) is $$r^2\frac{d\theta}{dt} = h \qquad (3)$$

where h is a constant. For convenience, set

$$r = \frac{1}{u} \quad (4) \qquad \text{and therefore} \qquad \frac{d\theta}{dt} = hu^2 \qquad (5)$$

Then $$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}$$

and therefore $$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2} \qquad (6)$$

Substituting (4), (5), (6) in (1) and reducing, we get

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \qquad (7)$$

The general solution of (7) can be expressed in the form

$$u = -c\cos(\theta - \epsilon) + \frac{\mu}{h^2}$$

where c and ϵ are constants. Therefore, since $r = 1/u$,

$$r = \frac{h^2/\mu}{1 - c(h^2/\mu)\cos(\theta - \epsilon)} \qquad (8)$$

and this is the equation, in polar coordinates, of a conic which has a focus at O, whose principal axis makes the angle ϵ with Ox, whose eccentricity is ch^2/μ, and whose latus rectum is $2h^2/\mu$ (p. 80, Ex. 6).

EXERCISE LIII

Solve the following equations:

1. $\dfrac{d^2y}{dx^2} = 1 + \left(\dfrac{dy}{dx}\right)^2$ **2.** $\dfrac{d^2y}{dx^2} - 2y\dfrac{dy}{dx} = 0$

3. A stone weighing m pounds is sent sliding along the ice with the initial velocity v_0 ft./sec. If the coefficient of friction between the stone and the ice is μ, the equation of the stone's motion is

$$m\, d^2x/dt^2 = -\, mg\mu$$

Show that the stone will come to rest when $t = v_0/g\mu$ and that the distance it has traveled will be $v_0^2/2\, g\mu$.

4. Show that, if under the earth's attraction and that force only a body P were to move toward the earth, its equation of motion would be $v\, dv/dx = -\, R^2g/x^2$, where R is the radius of the earth, g the attraction of gravity at the earth's surface, and x the distance of P from the earth's center. Show also that were P to start from rest at an infinite distance, it would reach the earth's surface with the velocity $\sqrt{2\, gR}$, about seven miles per second.

5. It can be shown that the attraction of a solid sphere S on a particle P within S is directed toward the center O of S, and is proportional to OP. Show that, were P to move in a straight tube passing through the earth's center O, its equation of motion would be $d^2x/dt^2 = -\, gx/R$, where $x = OP$. Show also that, if P were to start from rest at the earth's surface, it would reach the center in about $21\frac{1}{4}$ minutes.

6. A particle moving on Ox is attracted toward O by a force of magnitude k/x^3. If it starts from rest at $x = a$, how long will it be before it reaches O?

7. Discuss the motion of a particle whose initial velocity is v_0, under the action of a resistance proportional to the speed, and of this force only.

8. Referring to § 272, show that from the equation $r^2\, d\theta/dt = h$ it follows that the radius vector OP sweeps out equal areas in equal times.

9. Suppose that the path or orbit of P, § 272 (8), is an ellipse. The time T of a circuit of the orbit is then called the *periodic time*. If a and b are the major and minor semiaxes of the ellipse, its area is $\pi\, ab$ and its latus rectum is $2\, b^2/a$. Hence show that $T = 2\, \pi a^{3/2}/\mu^{1/2}$, and from this deduce Kepler's third law: In the motions of the planets about the sun, the squares of the periodic times of the several orbits are proportional to the cubes of their major axes.

10. Using the notation of § 272, find the general equation of the path of P on the supposition that $A_{OP} = -\, \mu/r^3$, $A_{PT} = 0$.

11. In case $A_{OP} = -k^2/r$, $A_{PT} = 0$, show that

$$\frac{d^2x}{dt^2} + k^2x = 0 \qquad \frac{d^2y}{dt^2} + k^2y = 0$$

and therefore that the projections of P on Ox and Oy are in simple harmonic motion. Show that the orbit of P is an ellipse with its center at O and that the periodic time is $2\pi/k$.

12. A particle P of mass m is constrained to move on a given curve. If P is under the action of a force whose x-, y-, z-components are X, Y, Z, and this force only, show that its equation of motion is

$$\frac{d^2s}{dt^2} = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}$$

13. Show that the equation of motion of a particle P of mass m moving under the action of gravity from A to B along a given curve C is $v\,dv/ds = g\,dy/ds$, and therefore that the speed which it will have at B is independent of the path.

14. A particle P of mass m which is attached to a fixed point O by a thin rod of length l (a simple pendulum) is swinging in a vertical circle. Show that its equation of motion is $l\,d^2\theta/dt^2 = -g\sin\theta$, where θ is the angle which OP makes with the vertical; therefore, that for small oscillations the motion is approximately simple harmonic.

ORDINARY DIFFERENTIAL EQUATIONS IN THREE VARIABLES

273. Exact equations. A differential equation in x, y, z of the form

$$P\,dx + Q\,dy + R\,dz = 0 \tag{1}$$

is called *exact* when its first member is the total differential of some function $u(x, y, z)$. The equation may then be written $du = 0$, and its general solution is $u = C$.

The necessary and sufficient condition that (1) *be an exact equation is that*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \qquad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \tag{2}$$

This can be proved by reasoning like that in § 253. Another proof that (2) is the sufficient condition is given in § 294.

274. Integrable equations. 1. We say that the equation

$$P\,dx + Q\,dy + R\,dz = 0 \tag{1}$$

is *integrable* if its first member is an exact differential or becomes one when multiplied by some integrating factor μ.

Thus, $yz\,dx + zx\,dy - xy\,dz = 0$ is integrable. For when multiplied by $1/xyz$ it becomes $dx/x + dy/y - dz/z = 0$, or $d \log xy/z = 0$, and has the solution $xy = Cz$.

2. *The necessary and sufficient condition that* (1) *be integrable is that* P, Q, R *satisfy the identity* [1]

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \equiv 0 \tag{2}$$

For if an integrating factor μ exists such that

$$\mu(P\,dx + Q\,dy + R\,dz) = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

then
$$\mu P = \frac{\partial u}{\partial x}, \qquad \mu Q = \frac{\partial u}{\partial y}, \qquad \mu R = \frac{\partial u}{\partial z}$$

and therefore

$$\frac{\partial}{\partial y}\mu P = \frac{\partial}{\partial x}\mu Q \quad \text{or} \quad P\frac{\partial \mu}{\partial y} + \mu\frac{\partial P}{\partial y} = Q\frac{\partial \mu}{\partial x} + \mu\frac{\partial Q}{\partial x}$$

$$\frac{\partial}{\partial z}\mu Q = \frac{\partial}{\partial y}\mu R \quad \text{or} \quad Q\frac{\partial \mu}{\partial z} + \mu\frac{\partial Q}{\partial z} = R\frac{\partial \mu}{\partial y} + \mu\frac{\partial R}{\partial y}$$

$$\frac{\partial}{\partial x}\mu R = \frac{\partial}{\partial z}\mu P \quad \text{or} \quad R\frac{\partial \mu}{\partial x} + \mu\frac{\partial R}{\partial x} = P\frac{\partial \mu}{\partial z} + \mu\frac{\partial P}{\partial z}$$

Multiplying these equations by R, P, Q, and adding, we get (2). Hence (2) is the necessary condition that (1) be integrable. In the next section it will be shown to be the sufficient condition also.

Example. Show that if P, Q, R satisfy (2), and λ be a function of x, y, z, then also

$$\lambda P\left(\frac{\partial}{\partial z}\lambda Q - \frac{\partial}{\partial y}\lambda R\right) + \lambda Q\left(\frac{\partial}{\partial x}\lambda R - \frac{\partial}{\partial z}\lambda P\right) + \lambda R\left(\frac{\partial}{\partial y}\lambda P - \frac{\partial}{\partial x}\lambda Q\right) \equiv 0 \tag{3}$$

[1] This may be expressed in the more easily remembered symbolic form

$$\begin{vmatrix} P & Q & R \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \equiv 0$$

275. The solution of integrable equations. When the condition (2) is satisfied, the solution of the equation (1) can be made to depend on solving equations in two variables.

EXAMPLE. Solve the equation

$$2\,xz\,dx + 2\,yz^2\,dy + (x^2 + 2\,y^2z - 1)\,dz = 0 \tag{1}$$

It will be found that the equation satisfies the condition § 274 (2). Were z a constant, (1) would reduce to

$2\,x\,dx + 2\,yz\,dy = 0$ (2) whose solution is $x^2 + y^2z = C$ (3)

Let us inquire whether if we regard C as a function of z instead of a constant, we can so determine it that (3) will represent a solution of (1).

Differentiating (3) with z variable,

$$2\,x\,dx + 2\,yz\,dy + y^2\,dz = dC \tag{4}$$

Multiplying (4) by z and subtracting from (1)

$$(x^2 + y^2z - 1)\,dz + z\,dC = 0 \tag{5}$$

which by aid of (3) reduces to

$$(C - 1)\,dz + z\,dC = 0 \tag{6}$$

and this involves C and z only and gives

$$(C - 1)z = c \quad \text{or} \quad C = c/z + 1 \tag{7}$$

Substituting (7) in (3), we obtain the solution of (1), namely

$$x^2 + y^2z = c/z + 1 \quad \text{or} \quad x^2z + y^2z^2 - z = c$$

The success of the method is due to the fact that by (3) we can eliminate both x and y from (5). The reason for this, as we shall see, is that (1) satisfies the condition § 274 (2).

In general, to solve any integrable equation

$$P\,dx + Q\,dy + R\,dz = 0 \tag{1}$$

first regard one variable, say z, as constant and solve

$$P\,dx + Q\,dy = 0 \tag{2}$$

Let the solution[1] be $v(x, y, z) = C$ (3)

[1] We assume that for the values of x, y, z, C under consideration the solution of (2) can be reduced to the form (3), by aid of the theorem of § 226.

Differentiate (3) completely regarding C as a function of z:

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \left(\frac{\partial v}{\partial z} - \frac{\partial C}{\partial z}\right)dz = 0 \tag{4}$$

Since (3) satisfies (2), a function $\lambda(x, y, z)$ exists such that

$$\frac{\partial v}{\partial x} = \lambda P \quad \text{and} \quad \frac{\partial v}{\partial y} = \lambda Q \tag{5}$$

Hence (4) is equivalent to (1) if C be such that

$$\frac{\partial v}{\partial z} - \frac{\partial C}{\partial z} = \lambda R \quad \text{or} \quad \frac{\partial v}{\partial z} - \lambda R = \frac{\partial C}{\partial z} \tag{6}$$

But, as will be shown immediately, when the condition of § 274 (2) is satisfied, (6) can by aid of (3) be reduced to an equation in C, z only. Let the solution of this reduced equation be $C = \phi(z) + c$. Then the solution of (1) is

$$v(x, y, z) = \phi(z) + c \tag{7}$$

It being supposed that the condition of § 274 (2) is satisfied, we are to prove that if (3) be solved for y, say, in terms of x, z, C, and the solution be substituted in (6), the result will involve only z and C, not x. But this is equivalent to showing that when x, z, C are the independent variables, and y the function of x, z, C defined by (3), the x-derivative of $\partial v/\partial z - \lambda R$ is 0, in other words, that

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial z} - \lambda R\right] + \frac{\partial}{\partial y}\left[\frac{\partial v}{\partial z} - \lambda R\right]\frac{\partial y}{\partial x} = 0 \quad \text{where} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} = 0$$

or that

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial z} - \lambda R\right]\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\left[\frac{\partial v}{\partial z} - \lambda R\right]\frac{\partial v}{\partial x} = 0 \tag{8}$$

But, using (5), the first member of (8) may be written

$$\left[\frac{\partial}{\partial z}\lambda P - \frac{\partial}{\partial x}\lambda R\right]\lambda Q - \left[\frac{\partial}{\partial z}\lambda Q - \frac{\partial}{\partial y}\lambda R\right]\lambda P$$

and by § 274 (3), this is identically equal to

$$\lambda R\left[\frac{\partial}{\partial y}\lambda P - \frac{\partial}{\partial x}\lambda Q\right] = \lambda R\left[\frac{\partial^2 v}{\partial y\,\partial x} - \frac{\partial^2 v}{\partial x\,\partial y}\right] = 0$$

276. Geometrical interpretation. Non-integrable equations. The equation

$$P\,dx + Q\,dy + R\,dz = 0 \tag{1}$$

is the condition that the directions whose direction ratios are $dx : dy : dz$ and $P : Q : R$ be perpendicular, § 212.

Hence if a point T be supposed to move in any manner such that at every position A the direction of the motion, given by $dx : dy : dz$, is perpendicular to the line through A whose direction ratios are $P_A : Q_A : R_A$, it will trace a curve K which represents a particular solution of (1). And the set of all such curves K will represent the general solution.

When (1) satisfies the condition § 274 (2), that is, is integrable, the curves K can be grouped so as to form the surfaces of some family of surfaces $u(x, y, z) = C$. On the other hand, when (1) is not integrable, no family of surfaces $u = c$ exists which contains all the curves K. In this case, however, as in the integrable case, on an arbitrarily chosen surface S there are infinitely many of the curves K, one through each point of S.

Example. Show that the equation $y\,dx - x\,dy - x\,dz = 0$ is not integrable, and find the integral curves K which are on the plane $z = x + y$.

Substituting $dz = dx + dy$ in $y\,dx - x\,dy - x\,dz = 0$, we get $(x - y)\,dx + 2\,x\,dy = 0$ whose solution is $(y + x)^2 = Cx$. The cylindrical surfaces represented by this equation intersect the plane $z = x + y$ in the required curves.

EXERCISE LIV

Solve the following equations:

1. $yz\,dx - xz\,dy - 2\,xy\,dz = 0$

2. $xz\,dx + yz\,dy = (z + 1)\,dz$

3. $x\,dx + z\,dz = (a^2 - x^2 - z^2)\,dy$

4. $dx + dy = (x + y + e^{-z})\,dz$

5. $(2\,x + 2\,xy + 2\,xz^2)\,dx = dy + 2\,z\,dz$

6. $x\,dx - (1 - x^2 - z^2)^{1/2}\,dy + z\,dz = 0$

7. $(x^2 + 2\,y^2 + 2\,z^2)\,dx + xy\,dy + xz\,dz = 0$

8. $(y + z)\,dx + (z + x)\,dy + (x + y)\,dz = 0$

9. $(ay - bz)\,dx + (cz - ax)\,dy + (bx - cy)\,dz = 0$

10. $2(y - z)\,dx + (x + 3\,y - 2\,z)\,dy - (x + y)\,dz = 0$

11. Show that the set of curves on the sphere $x^2 + y^2 + z^2 = a^2$ which satisfy the equation $3\,x\,dx + y(1 - x)\,dy + z\,dz = 0$ project into parabolas in the xy-plane.

SIMULTANEOUS EQUATIONS

277. Equations of the first order. The pair of equations

$$P\,dx + Q\,dy + R\,dz = 0 \quad P_1\,dx + Q_1\,dy + R_1\,dz = 0 \qquad (1)$$

gives

$$dx : dy : dz = (QR_1 - Q_1R) : (RP_1 - R_1P) : (PQ_1 - P_1Q) \qquad (2)$$

and therefore, by the reasoning in § 276, defines an infinite set of curves K, of which one will pass through any assigned point [1] A and will there have the direction given by (2) at A. The pair of equations (1) will be solved if two families of surfaces $u = c_1$, $v = c_2$ can be found whose curves of intersection are the curves K.

1. If one of the equations (1) is integrable, we may find $u = c_1$, $v = c_2$ as follows:

Example 1. Solve the pair of equations

$$dz = dx + dy \quad (1), \qquad z\,dy - x\,dz = 0 \quad (2)$$

The solution of (1) is $z = x + y + c_1$ (3). Substituting (3) in (2) gives the x, y equation $(y + c_1)\,dy - x\,dx = 0$, whose solution is $y^2 - x^2 + 2\,c_1y = c_2$ (4). In (4) replace c_1 by its value as given by (3); we obtain $2\,yz - x^2 - y^2 - 2\,xy = c_2$ (5). Hence the required solution is $z - x - y = c_1$, $2\,yz - x^2 - y^2 - 2\,xy = c_2$.

2. Let L, M, N denote the right members of (2), and λ, μ, ν any functions of x, y, z. Then

$$\frac{dx}{L} = \frac{dy}{M} = \frac{dz}{N} = \frac{\lambda\,dx + \mu\,dy + \nu\,dz}{\lambda L + \mu M + \nu N} \qquad (3)$$

Any pair of the equations (3) is equivalent to the given pair (1) or (2), and it may be easier to solve. It is sometimes possible to choose λ, μ, ν such that $\lambda L + \mu M + \nu N \equiv 0$. In that case, we must have $\lambda\,dx + \mu\,dy + \nu\,dz = 0$, and if

[1] Points are excepted where $QR_1 - Q_1R$, $RP_1 - R_1P$, $PQ_1 - P_1Q$ vanish simultaneously.

this equation be integrable its solution is one of the required equations $u = c_1$, $v = c_2$.

EXAMPLE 2. Solve $\quad dx : dy : dz = x(y - z) : y(z - x) : z(x - y)$

$$x(y - z) + y(z - x) + z(x - y) \equiv 0$$
$$\therefore dx + dy + dz = 0 \quad \therefore x + y + z = c_1 \tag{1}$$
$$yzx(y - z) + zxy(z - x) + xyz(x - y) \equiv 0$$
$$\therefore yz\,dx + zx\,dy + xy\,dz = 0 \quad \therefore xyz = c_2 \tag{2}$$

The integral curves K are those in which the planes (1) cut the surfaces (2).

278. Equations with constant coefficients. The following example illustrates the method of solving a pair of equations of the type

$$f_1(D)y + \phi_1(D)z = X_1 \qquad f_2(D)y + \phi_2(D)z = X_2$$

where D denotes d/dx; $f_1(D)$ and so on, polynomials in D with constant coefficients; and X_1, X_2 functions of x.

EXAMPLE. Solve

$$\frac{d^2y}{dx^2} + \frac{dz}{dx} + 6z = x, \quad \frac{dy}{dx} - y + 2z = x^2$$

The equations may be written:

$$D^2y + (D + 6)z = x \quad (1) \qquad (D - 1)y + 2z = x^2 \quad (2)$$

To eliminate y, we apply $D - 1$ to (1) and D^2 to (2), and subtract, the result being

$$(D^2 - 5D + 6)z = x + 1 \tag{3}$$

whose solution is $\quad z = c_1e^{2x} + c_2e^{3x} + \dfrac{x}{6} + \dfrac{11}{36} \qquad (4)$

Similarly, by eliminating z,

$$y = c'_1e^{2x} + c'_2e^{3x} - x^2 - \frac{5}{3}x - \frac{19}{18} \tag{5}$$

The constants c_1, c_2, c'_1, c'_2 are not independent. For substituting (4), (5) in (2), we get

$$(2c_1 + c'_1)e^{2x} + (2c_2 + 2c'_2)e^{3x} \equiv 0$$

$$\therefore 2c_1 + c'_1 = 0, \; c_2 + c'_2 = 0 \qquad \therefore c'_1 = -2c_1, \; c'_2 = -c_2.$$

EXERCISE LV

Solve the following pairs of equations :

1. $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{z}$

2. $\dfrac{dx}{x} = \dfrac{dy}{z} = -\dfrac{dz}{y}$

3. $\dfrac{dx}{x^2} = -\dfrac{dy}{xy} = -\dfrac{dz}{z^2}$

4. $\dfrac{dx}{bz - cy} = \dfrac{dy}{cx - az} = \dfrac{dz}{ay - bx}$

5. $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2\,xy} = \dfrac{dz}{2\,xz}$

6. $\dfrac{dx}{y + z} = \dfrac{dy}{z + x} = \dfrac{dz}{x + y}$

7. $\dfrac{dy}{dx} + 3\,y + 4\,z = 0,\quad \dfrac{dz}{dx} - 2\,y - 5\,z = 0$

8. $\dfrac{d^2x}{dt^2} = m^2y,\quad \dfrac{d^2y}{dt^2} = m^2x$

9. $\dfrac{dy}{dx} - \dfrac{dz}{dx} + y = \cos 2\,x,\quad \dfrac{d^2y}{dx^2} - \dfrac{dz}{dx} - 3\,y - z = e^{2x}$

10. Show that the differential equations of the curves orthogonal to the surfaces $F(x, y, z) = C$ are $dx : dy : dz = F_x : F_y : F_z$; and find the curves orthogonal to

1. $2\,x^2 + 4\,y^2 + z^2 = C$ 2. $xy + z = C$

PARTIAL DIFFERENTIAL EQUATIONS

279. Linear equations of the first order. Let x, y be independent variables, and z a variable dependent on x, y, and represent $\partial z/\partial x$ and $\partial z/\partial y$ by p and q. The equation

$$Pp + Qq = R \tag{1}$$

where P, Q, R denote functions of x, y, z, is said to be *linear*.

Suppose
$$z = f(x, y) \tag{2}$$
to be a particular solution of (1) and call the surface which it represents an *integral surface* of (1). The values of $p : q : -1$ at any point A of this surface (2) are the direction ratios of its normal at A, § 233 (5). Therefore since, by (1), we have $Pp + Qq + R(-1) = 0$, the normal to (2) at A is perpendicular to the direction whose ratios are the values of $P : Q : R$ at A. This consideration suggests the following method of finding the integral surfaces.

By solving the simultaneous equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{3}$$

as in § 277, we obtain a doubly infinite set of curves $u = c_1$, $v = c_2$ of which one, and generally but one, will pass through a given point A and have there for its direction ratios the values of $P : Q : R$ at A. From this set, select the singly infinite set S for which c_1 and c_2 satisfy any given relation $\phi(c_1, c_2) = 0$. Then the equation

$$\phi(u, v) = \phi(c_1, c_2) = 0 \tag{4}$$

will represent a surface on which all the curves of the set S lie; and the normal to this surface at any point A will be perpendicular to the curve of the set S through A, that is, to the direction whose ratios are the values of $P : Q : R$ at A. Hence the surface is an integral surface of (1).

We have therefore proved that every equation of the form $\phi(u, v) = 0$ is a solution of the given equation (1), and it can be shown conversely that, save in exceptional cases, every solution of (1) can be reduced to this form. Hence

The general solution of the linear equation (1) *is*

$$\phi(u, v) = 0 \tag{5}$$

where $u = c_1$, $v = c_2$ *denote any two independent integrals of the equations* (3), *and* ϕ *an arbitrary function subject only to the restriction that* ϕ_u, ϕ_v *exist.*

The curves $u = c_1$, $v = c_2$ are called the *characteristics* of the equation (1). Each particular solution of (1) represents a surface made up of these characteristics, and the general solution represents the set of all such surfaces.

One may obtain the integral surface of (1) which contains a given curve $g(x, y, z) = 0$, $h(x, y, z) = 0$, not a characteristic, by eliminating x, y, z between $g = 0$, $h = 0$, $u = c_1$, $v = c_2$. Let the result of the elimination be $f(c_1, c_2) = 0$; the required surface is $f(u, v) = 0$.

EXAMPLE 1. Find the general solution of $yp - xq = 0$.

The equations $\dfrac{dx}{y} = \dfrac{dy}{-x} = \dfrac{dz}{0}$ give $x^2 + y^2 = c_1$, $z = c_2$.

Hence the general solution is $\phi(x^2 + y^2, z) = 0$ or $x^2 + y^2 = \phi(z)$ which represents all surfaces of revolution about Oz as axis.

The characteristics are the ∞^2 set of circles $x^2 + y^2 = c_1$, $z = c_2$.

EXAMPLE 2. Find the integral surface of $2p - q = 1$ which contains the curve $y = x^2$, $z = x$ (1).

The equations $\dfrac{dx}{2} = \dfrac{dy}{-1} = \dfrac{dz}{1}$ give $x - 2z = c_1$, $y + z = c_2$ (2)

Eliminating x, y, z between (1) and (2) gives $c_2 + c_1 = c_1^2$

Hence the required surface is $(y + z) + (x - 2z) = (x - 2z)^2$.

EXERCISE LVI

Solve the following equations and discuss their characteristics:

1. $(y + z)p + (z + x)q = x + y$
2. $xp + z = 0$
3. $(bz - cy)p + (cx - az)q = ay - bx$
4. $yp + xq = z$
5. $xzp + yzq = xy$
6. $zp + yq - x = 0$
7. $xp + yq + xy - z = 0$
8. $x^2p - xyq + y^2 = 0$
9. $z = px + qy + x^2 + y^2$
10. $(x^2 - y^2)p + 2xyq = 0$

11. Show that $ap + bq = 1$ represents all cylinders whose elements have the direction ratios $a : b : 1$.

12. Show that $(x - a)p + (y - b)q = z - c$ represents all cones with vertex at (a, b, c).

13. Show that $xp + yq = 0$ represents all surfaces formed by lines which meet the z-axis and are parallel to the xy-plane.

14. Show that all surfaces for which the z-intercept of any tangent plane is twice the z of the point of contact satisfy the equation $px + qy + z = 0$. Find the surface of this type which cuts the plane $z = 1$ in the parabola $y = x^2$.

15. Find the differential equation of which $\phi(x - lz, y - mz) = 0$ is the general solution.

16. Show that the characteristics of $Pp + Qq = R$ are perpendicular to the surfaces, if any, represented by $Pdx + Qdy + Rdz = 0$.

17. Show that the integrating factors μ of $M\,dx + N\,dy = 0$ are integrals of

$$\frac{\partial \mu}{\partial x} N - \frac{\partial \mu}{\partial y} M + \mu \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = 0$$

280. Non-linear equations of the first order. Let

$$F(x, y, z, p, q) = 0 \tag{1}$$

denote an equation of a degree higher than one in p, q. Sometimes by inspection, or by methods too complicated to be considered here but which depend on the solution of certain linear equations, there can be obtained an equation of the form

$$f(x, y, z, c_1, c_2) = 0 \tag{2}$$

which satisfies (1). Here f denotes a definite function, and c_1, c_2 are arbitrary constants. It is customary to call (2) a *complete integral* of (1).

The equation (2) represents a doubly infinite set of surfaces. From this set select the singly infinite set for which c_2 is some given function of c_1, say $c_2 = \phi(c_1)$. The envelope of this subset, got, § 244, by eliminating c_1 between the equations

$$f[x, y, z, c_1, \phi(c_1)] = 0 \quad (3) \qquad \text{and } f_{c_1} = 0 \quad (4)$$

is a surface whose equation will also satisfy (1); for it will touch each of the surfaces (3) along the curve or curves in which that surface is cut by the corresponding surface (4). These curves are called *characteristics* of the equation (1).

When we interpret ϕ as an arbitrary function, that is, as any function for which the set of surfaces (3) has an envelope, then the equations (3), (4) represent the set of all surfaces which touch the surfaces (2) along their characteristics, and the pair (3), (4) is called the *general integral* of (1).

Finally a surface may exist each of whose points is an isolated point of contact with one of the surfaces (2). In that case its equation also will satisfy (1), and it is called the *singular integral* of (1). It may be found by eliminating c_1, c_2 between the equations

$$f = 0 \qquad f_{c_1} = 0 \qquad f_{c_2} = 0 \tag{5}$$

It can be shown (compare § 258) that the singular integral, if it exists, may also be found by eliminating p, q between the equations

$$F = 0 \qquad F_p = 0 \qquad F_q = 0 \tag{6}$$

There are infinitely many complete integrals, but generally speaking any complete integral with the corresponding general integral will include all the particular integrals of (1).

Example. Solve the equation $z = pq$.

By inspection, we find the complete integral

$$z = (x + a)(y + b) \tag{1}$$

where a, b are arbitrary constants. The corresponding general integral is

$$z = (x + a)\,[y + \phi(a)] \qquad y + \phi(a) + (x + a)\phi'(a) = 0 \tag{2}$$

The singular integral is $\qquad z = 0 \qquad (3)$

The equation also has the complete integral

$$4\,z = (cx + y/c + d)^2 \tag{4}$$

But the surfaces (4) are included in those represented by (2).

Thus (4) with $c = 1$, $d = 0$ and (2) with $\phi(a) = -\,a$ both give

$$4\,z = (x + y)^2 \tag{5}$$

281. Equations of higher order. There is a group of partial differential equations of the second order which play an important rôle in mathematical physics. The physicist is not concerned with general solutions of these equations but with those best adapted to the discussion of the problem at hand.

Example. Physical considerations show that in the steady flow of heat in a thin plate lying in the xy-plane the temperature u at any point (x, y) satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

Let the plate be a rectangle R of breath π and infinite length with its base on Ox and one of its sides on Oy. Suppose that the sides are kept at the temperature 0, and the base at the temperature 1; also that for interior points of R the temperature decreases as the distance from Ox increases, and approaches 0 as the distance approaches infinity.

We are required to find the solution of (1) which will satisfy these conditions and will give the temperature u at every interior point of R.

It is readily seen that (1) is satisfied by

$$u = e^{-ky} \sin kx \tag{2}$$

also, if k be any positive integer, that this u is 0 when $x = 0$ or π and that it $\to 0$ when $y \to \infty$. But it does not satisfy the condition $u = 1$ when $y = 0$ and $0 < x < \pi$.

To find a solution which will satisfy this last condition also, let us set

$$u = \sum_{k=0}^{\infty} A_k e^{-ky} \sin kx \tag{3}$$

where the A_k's denote undetermined constants. Since every term of this series satisfies (1), the same is true of the series itself if, as we shall assume, it converges. It also satisfies the conditions $u = 0$, when $x = 0$ or π, and $u \to 0$ when $y \to \infty$. To satisfy the remaining condition, $u = 1$ when $y = 0$ and $0 < x < \pi$, the A_k's must be such that

$$1 = \sum_{k=0}^{\infty} A_k \sin kx \tag{4}$$

for all values of x between 0 and π. As will be shown later in the chapter on Fourier series, we can in fact determine the A_k's so as to satisfy this requirement. When the values so determined are substituted in (3) it becomes the required solution.

EXERCISE LVII

1. Show that the general solution of $\dfrac{\partial^2 z}{\partial x\, \partial y} = 0$ is $z = \phi(x) + \psi(y)$ where ϕ and ψ denote arbitrary functions.

2. Find the general solutions of the following equations

$$\frac{\partial^2 z}{\partial x\, \partial y} = x + y + \frac{y}{x} \qquad\qquad \frac{\partial^2 z}{\partial x^2} = x^2 y$$

3. Show that, if x, y are the independent variables, the general solution of $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$ is $z = \phi(y)e^{ax} + \psi(y)e^{-ax}$, where ϕ and ψ denote arbitrary functions.

4. Show that the equation with constant coefficients

$$a \frac{\partial^2 z}{dx^2} + b \frac{\partial^2 z}{\partial x\, \partial y} + c \frac{\partial^2 z}{\partial y^2} = 0$$

is satisfied by $z = e^{kx+ly}$ if k, l satisfy the equation $ak^2 + bkl + cl^2 = 0$. Show that the like is true of any partial differential equation with constant coefficients.

5. Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ has the solution $u = \phi(x + iy) + \psi(x - iy)$ where ϕ and ψ denote arbitrary functions, and $i = \sqrt{-1}$.

6. Show that $\frac{\partial^2 z}{\partial t^2} - a^2 \frac{\partial^2 z}{\partial x^2} = 0$, the equation of a vibrating cord, has the solution $z = \phi(x + at) + \psi(x - at)$, where ϕ and ψ are arbitrary functions.

7. Show that $z = px + qy + f(p, q)$ has the complete integral $z = ax + by + f(a, b)$. What is the corresponding general integral and what the singular integral when $f(p, q) = pq$?

8. Show that any equation of the form $f(p, q) = 0$ has the complete integral $z = ax + by + c$, where $f(a, b) = 0$. Show that the surfaces represented by the corresponding general integral are developable surfaces, § 244.

9. Show that every developable surface satisfies some equation of the form $f(p, q) = 0$ and therefore that all developable surfaces satisfy the equation

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 = 0$$

10. Find developable surfaces for which p, q satisfy each of the following conditions:

$$p^2 + q^2 = 1 \qquad\qquad pq = p + q$$

XXVI. LINE AND SURFACE INTEGRALS

THEOREMS RESPECTING DEFINITE INTEGRALS

282. Mean value theorem for double and triple integrals. As in § 149, let S denote a given region in the xy-plane and also its area.

1. *If $f(x, y)$ is continuous in S, there is a point $(\bar{x}, \bar{y})$ in S such that*

$$\int_S f(x, y)\, dS = f(\bar{x}, \bar{y}) \cdot S \tag{1}$$

For by the definition of $\int_S f(x, y)\, dS$, § 149, its value is between the least and greatest values of $f(x, y) \cdot S$ in S. Therefore, § 18, 2, since $f(x, y) \cdot S$ is continuous in S, there is a point $(\bar{x}, \bar{y})$ in S such that $f(\bar{x}, \bar{y}) \cdot S = \int_S f(x, y)\, dS$.

The like is true of integrals of the type $\int_S f(x, y, z)\, dS$.

2. *If f is continuous, and ϕ is both continuous and positive in S, there is a point P in S such that, f_P denoting the value of f at P,*

$$\int_S f\phi\, dS = f_P \int \phi\, dS \tag{2}$$

For if M and m denote the greatest and least values of f in S, then

$$\int_S (M - f)\phi\, dS > 0 \quad \therefore M\int_S \phi\, dS > \int f\phi\, dS \tag{3}$$

$$\int_S (f - m)\phi\, dS > 0 \quad \therefore \int_S f\phi\, dS > m\int_S \phi\, dS \tag{4}$$

and (2) follows from (3) and (4), as in the proof of 1.

Example. Extend the theorem of § 145 to double and triple integrals.

283. Differentiation under the integral sign. Let $f(x, \alpha)$ be a continuous function of x and the parameter α in the region R in which x is between a and b and α between A and B. Suppose also that $f_\alpha(x, \alpha)$ exists and is continuous in R.

Evidently $\int_a^b f(x, \alpha)\, dx$ is a function of α. We are to prove that

$$\textit{If } F(\alpha) = \int_a^b f(x, \alpha)\, dx \qquad \textit{then} \qquad F'(\alpha) = \int_a^b f_\alpha(x, \alpha)\, dx$$

For $\quad F(\alpha + \Delta\alpha) - F(\alpha) = \int_a^b [f(x, \alpha + \Delta\alpha) - f(x, \alpha)]\, dx$

Hence, applying the mean value theorem, and then dividing by $\Delta\alpha$,

$$\frac{F(\alpha + \Delta\alpha) - F(\alpha)}{\Delta\alpha} = \int_a^b f_\alpha(x, \alpha + \theta\,\Delta\alpha)\, dx \tag{1}$$

Set $\quad f_\alpha(x, \alpha + \theta\,\Delta\alpha) = f_\alpha(x, \alpha) + \eta$

Since f_α is uniformly continuous in R, § 148 (5), if any positive number ϵ be assigned we can find another positive number δ such that everywhere in R, when $|\Delta\alpha| < \delta$ we shall have $|\eta| < \epsilon$, and therefore $|\int_a^b \eta\, dx| < \epsilon\, |b - a|$. Hence it follows from (1), when $\Delta\alpha \to 0$, that

$$F'(\alpha) = \int_a^b f_\alpha(x, \alpha)\, dx \tag{2}$$

If the limits a and b are themselves functions of α, then, §§ 126, 217,

$$F'(\alpha) = \int_a^b f_\alpha(x, \alpha)\, dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha} \tag{3}$$

Example 1. Show that $v = \int_0^{2\pi} e^{k(x\cos\theta + y\sin\theta)}\, \phi(\theta)\, d\theta$ is a solution of

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = k^2 v$$

Example 2. Verify (3) for the integral $\int_\alpha^{2\alpha}(3\,x^2 + 2\,\alpha x + \alpha^2)\, dx$.

284. The Gamma function. By the tests indicated in Examples 26 and 27 on p. 152, it is readily shown that the integral $\int_0^\infty x^{n-1}e^{-x}\, dx$ exists when $n > 0$. Its value is a function of the parameter n. This function is called the *gamma function* and is represented by $\Gamma(n)$. Hence

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\, dx \qquad n > 0 \tag{1}$$

If we integrate $\int_{x_1}^{x_2} x^n e^{-x}\, dx$ by parts, and in the result let $x_1 \to 0$ and $x_2 \to \infty$, we obtain $\int_0^\infty x^n e^{-x}\, dx = n\int_0^\infty x^{n-1}e^{-x}\, dx$. Hence

$$\Gamma(n + 1) = n\Gamma(n) \tag{2}$$

By (1), we have $\Gamma(1) = 1$. Hence (2) gives successively $\Gamma(2) = 1$, $\Gamma(3) = 2 \cdot 1$, and in general, if n be any positive integer greater than one, $\Gamma(n) = (n-1)!$.

By a similar procedure, we may express the value of $\Gamma(n)$ for any positive non-integral n in terms of its value for an n between 1 and 2; and the values of $\Gamma(n)$ for n between 1 and 2 have been tabulated.

The definition (1) applies only when n is positive. But we may write (2) in the form

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \tag{3}$$

and then use it to define $\Gamma(n)$ for negative values of n.

For when n is between -1 and 0, then $n+1$ is between 0 and 1, and therefore $\Gamma(n+1)$ is known by (1); hence (3) gives us $\Gamma(n)$. In like manner the values of $\Gamma(n)$ for n between -2 and -1 are determined by its values for n between -1 and 0; and so on. Setting $n = 0$, -1, -2, $\cdots$ successively, we find that the definition (3) makes $\Gamma(n)$ infinite for 0 and negative integral values of n.

JACOBIANS

285. Properties of functional determinants or Jacobians. Let u, v be independent variables referred to rectangular axes Ou, Ov, and let x, y be given functions of u, v in some region D of the uv-plane:

$$x = \phi(u, v) \qquad\qquad y = \psi(u, v) \tag{1}$$

ϕ, ψ and their first partial derivatives being continuous.

Represent the functional determinant, § 229, or Jacobian, J, of ϕ and ψ, or x and y, with respect to u and v, namely

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \qquad \text{by} \qquad \frac{\partial(x, y)}{\partial(u, v)} \tag{2}$$

The Jacobian has properties analogous to those of the derivative.

1. Suppose that u, v are themselves functions of independent variables r, s, these functions possessing continuous partial derivatives. Then

$$\frac{\partial(x, y)}{\partial(r, s)} = \frac{\partial(x, y)}{\partial(u, v)} \frac{\partial(u, v)}{\partial(r, s)} \tag{3}$$

For
$$\begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial s} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u}\dfrac{\partial u}{\partial r} + \dfrac{\partial x}{\partial v}\dfrac{\partial v}{\partial r} & \dfrac{\partial x}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial x}{\partial v}\dfrac{\partial v}{\partial s} \\ \dfrac{\partial y}{\partial u}\dfrac{\partial u}{\partial r} + \dfrac{\partial y}{\partial v}\dfrac{\partial v}{\partial r} & \dfrac{\partial y}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial y}{\partial v}\dfrac{\partial v}{\partial s} \end{vmatrix}$$

which is the product of the two determinants $\partial(x, y)/\partial(u, v)$ and $\partial(u, v)/\partial(r, s)$, § 203.

2. Let (u_1, v_1) be any point of D where the Jacobian (2) is not 0, and let (x_1, y_1) be the corresponding point in the xy-plane given by (1) and § 230. Then u, v are one-valued continuous functions of x, y:

$$u = \Phi(x, y) \qquad\qquad v = \Psi(x, y) \tag{4}$$

in the neighborhood of (x_1, y_1) and take the values u_1, v_1 at (x_1, y_1), § 230.

The pair of functions (4) is called the *inverse* of the pair (1). By setting $r = x$, $s = y$ in (3), we obtain

$$1 = \frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} \qquad \therefore \qquad \frac{\partial(x, y)}{\partial(u, v)} = 1 \div \frac{\partial(u, v)}{\partial(x, y)} \tag{5}$$

3. Let x, y be functions of the three variables r, s, t:

$$x = \phi(r, s, t) \qquad\qquad y = \psi(r, s, t) \tag{6}$$

and let r, s, t in turn be functions of the independent variables u, v.

The Jacobian J of x and y with respect to u and v is given by the formula

$$J = \frac{\partial(x, y)}{\partial(r, s)} \frac{\partial(r, s)}{\partial(u, v)} + \frac{\partial(x, y)}{\partial(s, t)} \frac{\partial(s, t)}{\partial(u, v)} + \frac{\partial(x, y)}{\partial(t, r)} \frac{\partial(t, r)}{\partial(u, v)} \tag{7}$$

For

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{array}{cccccc} 1 & 2 & 3 & 1' & 2' & 3' \end{array}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial r}\dfrac{\partial r}{\partial u} + \dfrac{\partial x}{\partial s}\dfrac{\partial s}{\partial u} + \dfrac{\partial x}{\partial t}\dfrac{\partial t}{\partial u} & \dfrac{\partial x}{\partial r}\dfrac{\partial r}{\partial v} + \dfrac{\partial x}{\partial s}\dfrac{\partial s}{\partial v} + \dfrac{\partial x}{\partial t}\dfrac{\partial t}{\partial v} \\ \dfrac{\partial y}{\partial r}\dfrac{\partial r}{\partial u} + \dfrac{\partial y}{\partial s}\dfrac{\partial s}{\partial u} + \dfrac{\partial y}{\partial t}\dfrac{\partial t}{\partial u} & \dfrac{\partial y}{\partial r}\dfrac{\partial r}{\partial v} + \dfrac{\partial y}{\partial s}\dfrac{\partial s}{\partial v} + \dfrac{\partial y}{\partial t}\dfrac{\partial t}{\partial u} \end{vmatrix}$$

Resolve this determinant into a sum of determinants with simple columns by the theorem of § 199, 7. and in this sum represent the determinant with the columns marked 1 and 1′ by the symbol (1, 1′), and so on. It is easily shown that the determinants (1, 1′), (2, 2′), (3, 3′) are 0, and that the sums

$$(1, 2') + (2, 1') \qquad (2, 3') + (3, 2') \qquad (3, 1') + (1, 3')$$

are equal to the first, second and third terms of (7) respectively.

4. The functional determinant or Jacobian of any number of functions $x_1, x_2, \cdots, x_n$, of the same number of variables $u_1, u_2, \cdots, u_n$, is represented by the symbol

$$\frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(u_1, u_2, \cdots, u_n)}$$

The theorems in 1., 2., 3. hold good for any such Jacobian.

LINE AND SURFACE INTEGRALS

286. Line integrals. Let AB be a curve arc which is met by parallels to Ox or Oy in single points, and on which y is a continuous function of x, and x of y; AB having an equation of the form $y = \phi(x)$, or $x = \psi(y)$, where ϕ and ψ denote one-valued, continuous functions.

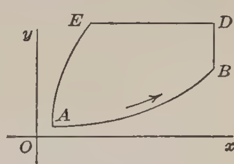

FIG. 127.

And let $f(x, y)$ be a function of x, y which is continuous on the arc AB.

It being also supposed that $\phi(x)$ and $\psi(y)$ have continuous derivatives, think of AB as traced by a point P which starts at A, and regard s, the length of the arc AP, as the independent variable.

Divide the arc AB into n parts all of which $\to 0$ when $n \to \infty$. Let δs denote any one of these parts; (x, y), any point of δs; and δx, δy, the projections of δs on Ox, Oy.

When $n \to \infty$ the sum $\Sigma f(x, y)\, \delta x$ approaches a limit. For

$$\Sigma f(x, y)\, \delta x = \Sigma f[x, \phi(x)]\, \delta x \to \int_a^b f[x, \phi(x)]\, dx$$

where a and b are the abscissas of A and B. We call $\lim \Sigma f(x, y)\, \delta x$ the x-integral of $f(x, y)$ along AB, and represent it by $\int_{AB} f\,(x, y)\, dx$.

In like manner, by definition,

$$\int_{AB} f(x, y)\, dy = \lim \Sigma f(x, y)\, \delta y$$

Evidently

$$\int_{BA} f\, dx = -\int_{AB} f\, dx \quad \text{and} \quad \int_{BA} f\, dy = -\int_{AB} f\, dy$$

We extend these definitions to line segments, like BD or DE, parallel to Oy or Ox. Since the projection of any part δs of BD on Ox is 0, $\int_{BD} f\, dx = 0$. Similarly $\int_{DE} f\, dy = 0$.

Finally let the point P trace any path C which can be separated into a finite number of non-intersecting pieces of the types AB, BD, DE. We define $\int_C f\, dx$ and $\int_C f\, dy$ as the algebraic sums of the corresponding integrals along the several pieces. If we reverse the sense in which C is traced, we merely change the signs of $\int_C f\, dx$ and $\int_C dy$. When C, like $ABDEA$ in Fig. 127, is the contour of a closed region S, we take as the positive sense of motion along C that for which the tracing point has S at its left.

Let P and Q be any two functions of x, y, both continuous on C. The sum of $\int_C P\, dx$ and $\int_C Q\, dy$ is written

$$\int_C P\, dx + Q\, dy,$$

and this is taken as the general expression for a line integral in x, y. Observe that if v_T denote the projection of the vector whose x- and y-components are P and Q, on the tangent to the curve C, we have $\int_C P\, dx + Q\, dy = \int_C v_T\, ds$.

We have analogous definitions of integrals along space curves.

Example 1. If AB is the arc of the parabola $y = x^2$ between $A(0, 0)$ and $B(1, 1)$, then

$$\int_{AB} (x + y)\, dx + x^2\, dy = \int_0^1 (x + x^2)\, dx + \int_0^1 y\, dy = 4/3$$

EXAMPLE 2. If C consists of the line segments AB and BD joining $A(a, b)$, $B(x, b)$ and $D(x, y)$, then

$$\begin{aligned}\textstyle\int_C y\,dx + x\,dy &= \textstyle\int_{AB} y\,dx + \int_{BD} x\,dy = \int_a^x b\,dx + \int_b^y x\,dy \\ &= b(x-a) + x(y-b) = xy - ab\end{aligned}$$

287. Green's formula. 1. Let S denote any region, bounded by a single contour C, which can be divided into pieces like $S_1 = KLMN$ in Fig. 128, KL and MN being non-intersecting curve arcs of the type described in § 286. And let P and Q be any functions of x, y which, together with $\partial P/\partial y$ and $\partial Q/\partial x$ are continuous in S and on C. We have

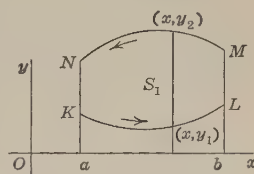

FIG. 128.

$$\begin{aligned}\iint_{S_1} \frac{\partial P}{\partial y}\,dy\,dx &= \int_a^b \int_{y_1}^{y_2} \frac{\partial P}{\partial y}\,dy\,dx \\ &= \int_a^b P(x, y_2)\,dx - \int_a^b P(x, y_1)\,dx \\ &= -\int_{MN} P(x, y)\,dx - \int_{KL} P(x, y)\,dx \qquad (1)\end{aligned}$$

the line integrals in the last member being taken in the positive sense along the parts of the contour C of S which belong to S_1. Hence by adding the equations (1) for all the parts S_1 of S, we have

$$\iint_S \frac{\partial P}{\partial y}\,dy\,dx = -\int_C P\,dx \qquad (2)$$

We can show in like manner that

$$\iint_S \frac{\partial Q}{\partial x}\,dx\,dy = \int_C Q\,dy \qquad (3)$$

and subtracting (2) from (3) we obtain

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_C P\,dx + Q\,dy \qquad (4)$$

This identity is called *Green's formula*. It has many important applications. Observe that (2) and (3) are particular cases of it.

If in (4) we set $P = y, Q = 0$; $P = 0, Q = x$; $P = -y$, $Q = x$, we obtain the formulas:

$$\text{Area } S = -\int_C y\,dx = \int_C x\,dy = \tfrac{1}{2}\int_C x\,dy - y\,dx \qquad (5)$$

2. *If $\partial Q/\partial x = \partial P/\partial y$ throughout S, then*

$$\int_C P\,dx + Q\,dy = 0 \qquad (6)$$

Conversely, if along every closed curve in a certain region R the integral $\int_C$ in (4) is 0, then $\partial Q/\partial x - \partial P/\partial y$ is 0 throughout R. For otherwise, $\partial Q/\partial x$ and $\partial P/\partial y$ being continuous, there would be a region in R in which the sign of $\partial Q/\partial x - \partial P/\partial y$ was constant, and, by (4), the integral $\int_C$ along the contour of this region would not be 0.

3. The formula (4) may be extended to regions bounded by more than one contour.

Thus consider the region S between the contours C_1 and C_2 in Fig. 129. Make the cut ad and so transform S into a region S' bounded by the single contour $C = abcadefda$. By (4) we have $\iint_S = \iint_{S'} = \int_C$. But the parts of $\int_C$ along ad and da cancel each other. Hence $\iint_S = \int_{C_1} + \int_{C_2}$, where both C_1 and C_2 are supposed traced by a point having S at its left.

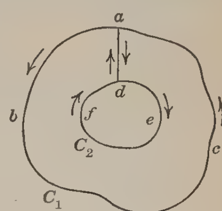

FIG. 129.

If $\partial Q/\partial x = \partial P/\partial y$ throughout S, and both C_1 and C_2 be supposed traced clockwise, then $\int_{C_1} = \int_{C_2}$.

4. A region in which no closed curve can be drawn that encloses a boundary point of the region is said to be *simply connected;* one in which such a curve can be drawn, *multiply connected.* Thus in Fig. 129 the region bounded by C_1 and C_2 is multiply connected; that bounded by C_1, C_2 and ad is simply connected.

288. The function $F(x, y) = \int_{a,b}^{x,y} P\,dx + Q\,dy$. 1. Suppose that throughout a region R, of the kind described in § 214, 2, we have $\partial Q/\partial x = \partial P/\partial y$. Then if A and B are any two points of R, the value of $\int_{AB} P\,dx + Q\,dy$ is the

same for all paths in R from A to B and may be represented by $\int_A^B P\,dx + Q\,dy$.

For if ADB and AEB denote any two such paths, and (ADB), (AEB) the values of $\int_{AB} P\,dx + Q\,dy$ along them, then, by § 287 (6),

$$(ADB) + (BEA) = 0$$

$$\therefore\ (ADB) = -(BEA) = (AEB).$$

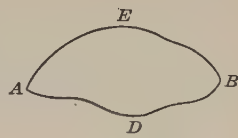

FIG. 130.

This proof does not apply to all paths in a region like S in Fig. 129. Let C' be a curve in S that encloses C_2. It cannot be inferred that $\int_{C'} P\,dx + Q\,dy$ is 0 unless it be known that P, Q, $\partial Q/\partial x$, $\partial P/\partial y$ are continuous and $\partial Q/\partial x = \partial P/\partial y$, not only in S, but also in the entire region interior to C_2.

2. Let (a, b) denote a fixed point, and (x, y) a variable point, both in R. By 1., $\int_{a,b}^{x,y} P\,dx + Q\,dy$ has the same value for all paths in R from (a, b) to (x, y); it is therefore a function of (x, y) only.

If $F(x, y) = \displaystyle\int_{a,b}^{x,y} P\,dx + Q\,dy$, *then* $\dfrac{\partial F}{\partial x} = P$ *and* $\dfrac{\partial F}{\partial y} = Q$.

For give x the increment Δx. It then follows from 1. that

$$F(x + \Delta x, y) - F(x, y) = \int_{x,y}^{x+\Delta x, y} P\,dx + Q\,dy$$

But $\int_{x,y}^{x+\Delta x, y} P\,dx = P(x + \theta\,\Delta x, y)\Delta x$, and $\int_{x,y}^{x+\Delta x, y} Q\,dy = 0$

Hence $$\frac{\partial F}{\partial x} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x, y) - F(x, y)}{\Delta x} = P(x, y)$$

Similarly $$\frac{\partial F}{\partial y} = Q(x, y)$$

Hence $P\,dx + Q\,dy$ is the differential of $F(x, y)$, and we have another proof that $P\,dx + Q\,dy$ is an exact differential if $\partial Q/\partial x = \partial P/\partial y$.

If $u(x, y)$ be any known integral of $P\,dx + Q\,dy$, then, by § 223, 2.,

$$F(x, y) = \int_{a,b}^{x,y} P\,dx + Q\,dy = u(x, y) + C$$

Let $x, y \to a, b$. We get

$$0 = u(a, b) + C \quad \therefore C = -u(a, b)$$

Hence $$F(x, y) = u(x, y) - u(a, b)$$

We can find such an integral $u(x, y)$ by the formula

$$\int_{x_0, y_0}^{x, y} P\,dx + Q\,dy = \int_{x_0}^{x} P(x, y_0)\,dx + \int_{y_0}^{y} Q(x, y)\,dy \qquad (1)$$

289. Transformation of double integrals. Into what integral is $\iint_S f(x, y)\,dx\,dy$ transformed by the substitution

$$x = \phi(u, v) \qquad y = \psi(u, v) \qquad (1)$$

where u, v denote new independent variables?

Regarding u, v as variables referred to rectangular axes, O_1u, O_1v, we assume that there is a region S' in the uv-plane to which the equations (1) make the given xy-region S correspond in such a manner that to each interior point (u, v) of S' there corresponds an interior point (x, y) of S, and but one, and that when (u, v) traces the contour C' of S', then (x, y) traces the contour C of S in the same or the opposite sense. We also suppose that in S' and on C' the functions ϕ and ψ and their first and second partial derivatives are continuous, and that

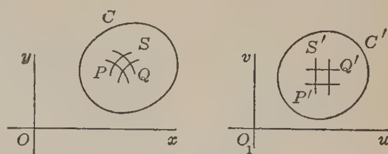

FIG. 131.

$$J = \frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial v} - \frac{\partial\phi}{\partial v}\frac{\partial\psi}{\partial u} \qquad (2)$$

does not vanish, and is therefore of constant sign. We shall prove that

$$\iint_S f(x, y)\,dx\,dy = \iint_{S'} f(\phi, \psi)\,|J|\,du\,dv \qquad (3)$$

To a division of S' into small rectangles $\delta S'(P'Q')$ by sets of lines $u = a$ and $v = b$ parallel to O_1v and O_1u corresponds the division of S into small curvilinear quadrilaterals $\delta S(PQ)$ by the sets of curves

$x = \phi(a, v)$, $y = \psi(a, v)$ and $x = \phi(u, b)$, $y = \psi(u, b)$. By § 149, the given integral $\iint_S f(x, y)\, dx\, dy$ equals $\lim \Sigma f(x, y)\, \delta S$. Hence to prove our theorem we must find the expression for δS in terms of u, v.

1. We begin by finding the expression for the area S in terms of u, v. By § 287 (5), we have $S = \int_C x\, dy$, the integral being taken in the positive sense along C. But, by hypothesis, (x, y) will trace C in this sense when (u, v) traces C' in one sense or the other. Hence if ds' denote the differential of arc of C', we have, by §§ 132, 107,

$$S = \int_C x\, dy = \pm \int_{C'} \phi \frac{d\psi}{ds'}\, ds' = \pm \int_{C'} \phi \frac{\partial \psi}{\partial u}\, du + \phi \frac{\partial \psi}{\partial v}\, dv \tag{4}$$

the sign being $+$ or $-$ according as C' is traced in the positive or negative sense. But by Green's formula, § 287 (4), with x, y, P, Q replaced by u, v, $\phi\, \partial\psi/\partial u$, $\phi\, \partial\psi/\delta v$, the last member of (4) reduces to

$$\pm \iint_{S'} \left[\frac{\partial}{\partial u}\left(\phi \frac{\partial \psi}{\partial v}\right) - \frac{\partial}{\partial v}\left(\phi \frac{\partial \psi}{\partial u}\right)\right] du\, dv = \pm \iint_{S'} J\, du\, dv \tag{5}$$

Since the area S is positive and by hypothesis the sign of J is constant in S', it follows from (4) and (5) that $\pm J = |J|$ and therefore that

$$S = \iint_{S'} |J|\, du\, dv \tag{6}$$

2. It follows from (6) and § 282 (1), that the area of the part δS of S corresponding to the part $\delta S'$ of S' can be expressed in the form

$$\delta S = |J(\bar{u}, \bar{v})|\, \delta S' \tag{7}$$

where $(\bar{u}, \bar{v})$ denotes some point of $\delta S'$. Hence, if $(\bar{x}, \bar{y})$ be the corresponding point of δS, we have

$$f(\bar{x}, \bar{y})\, \delta S = f[\phi(\bar{u}, \bar{v}), \psi(\bar{u}, \bar{v})]\, |J(\bar{u}, \bar{v})|\, \delta S'$$

and since $\iint_S f(x, y)\, dx\, dy = \lim \Sigma f(\bar{x}, \bar{y})\, \delta S,$

we have finally $\iint_S f(x, y)\, dx\, dy = \iint_{S'} f(\phi, \psi)\, |J|\, du\, dv$

Example. If $x = r \cos \theta$, $y = r \sin \theta$, show that $J = r$, therefore that

$$\iint_S f(x, y)\, dx\, dy = \iint_{S'} f(r \cos \theta, r \sin \theta)\, r\, dr\, d\theta$$

290. Surface area. Let S denote a piece of a surface bounded by a single contour L, and let D denote the projection of S on some plane E, S and E being such that no normal to E meets S in more than one point and that no normal to S is parallel to E.

Referred to rectangular axes O-xyz in which E is the xy-plane, S has an equation of the form $z = f(x, y)$. We

suppose S to be such that $f(x, y)$, $p = f_x$, $q = f_y$ are continuous in D, — which means that S has a definite tangent plane at every point and that this plane moves continuously when the point of tangency moves continuously on S.

Divide D into parts of the type described in § 149, and by cylindrical surfaces through their boundaries and parallel to Oz make the corresponding division of S. Let δD denote any one of these parts of D and also its area; and let δS denote the corresponding part of S, and δT the area of that part of any tangent plane to δS which like δS projects into δD. The angle which the tangent plane makes with the xy-plane E is the same as the acute angle γ which the corresponding normal to δS makes with Oz. Hence $\delta D = \delta T \cos \gamma$, and therefore, § 233 (5),

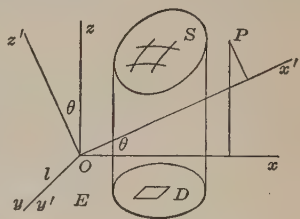

FIG. 132.

$$\delta T = \delta D / \cos \gamma = (p^2 + q^2 + 1)^{1/2} \, \delta D$$

By § 149, since $(p^2 + q^2 + 1)^{1/2}$ is continuous in D, the sum

$$\Sigma \, \delta T = \Sigma \, \delta D / \cos \gamma = \Sigma (p^2 + q^2 + 1)^{1/2} \, \delta D$$

will approach a limit when all the parts $\delta D \to 0$. This limit, $\int_D \sec \gamma \, dD$, is called the *area of S*. Hence if we make the division of D into parts by parallels to Ox, Oy, so that $\delta D = \delta x \, \delta y$, we have, by definition,

$$\text{area } S = \iint_D \frac{dx \, dy}{\cos \gamma} = \iint_D (p^2 + q^2 + 1)^{1/2} \, dx \, dy \qquad (1)$$

The value of area S given by (1) is evidently independent of the choice of the xy-axes in the plane E. It is also independent of the choice of the plane E itself.[1] For let E' denote a second plane, l its line of intersection with E, and θ the angle which E' makes with E. Take any point O of l as origin and l as y-axis in both planes, the axes O-xyz and O'-$x'y'z'$ then lying as in Fig. 132. Finally let D' denote the pro-

[1] E. B. Wilson, Advanced Calculus, p. 339.

jection of S on E', and γ' the acute angle which a normal to S makes with Oz'. Then

$$\iint_D \frac{dx\,dy}{\cos\gamma} = \iint_{D'} \frac{dx'\,dy'}{\cos\gamma(\cos\theta + p\sin\theta)} = \iint_{D'} \frac{dx'\,dy'}{\cos\gamma'} \qquad (2)$$

For, as the figure shows, the x, y and x', y' coordinates of any point of S are connected by the relations

$$y' = y \qquad x' = x\cos\theta + z\sin\theta, \text{ where } z = f(x, y) \qquad (3)$$

Hence $\qquad dx = [1/(\cos\theta + p\sin\theta)]\,dx' \quad \text{and} \quad dy = dy' \qquad (4)$

The substitution (3), (4) transforms the first integral in (2) into the second, the two integrals being equal by § 132 since only one variable, x, is changed. But the denominator in the second integral equals $\cos\gamma'$. For the direction ratios of a normal to S, referred to the axes $O\text{-}xyz$, are $-p : -q : 1$, and those of Oz' are $-\sin\theta : 0 : \cos\theta$. Hence, § 211 (1),

$$\cos\gamma' = \frac{p\sin\theta + \cos\theta}{(p^2 + q^2 + 1)^{1/2}} = \cos\gamma(p\sin\theta + \cos\theta)$$

EXERCISE LVIII

1. Find $\int_C x^2\,ds$ when C is the circle $x = a\cos\theta$, $y = a\sin\theta$.

2. Find $\int_C (x^2 + y)\,dx + (2x + y^2)\,dy$ when C is the square whose angular points are (1, 1), (2, 1), (2, 2), (1, 2). Verify by aid of Green's formula.

3. Given the points $A(2, 1)$, $B(6, 1)$, $C(6, 3)$, compute the integral $\int_{2,1}^{6,3} y^2\,dx + x^2\,dy$ along the straight paths ABC and AC.

4. Compute $\int_{0,0}^{1,-1} (x + 2y)\,dx + yx\,dy$ along the paths $y = -x^2$, and $x = t^2$, $y = t^3$.

5. Show that the value of $\int_{1,2}^{4,5} 2xy\,dx + (x^2 + 2y)\,dy$ is independent of the path, and compute it.

6. Show that $\dfrac{x^2 - y^2}{x^2 y}\,dx + \dfrac{y^2 - x^2}{xy^2}\,dy$ is an exact differential, and find its integral by aid of the formula § 288 (1).

7. Show that $\displaystyle\int_C \frac{x\,dy - y\,dx}{x^2 + y^2}$ is 0 if C is any circle which does not meet or enclose the origin; is 2π if C is the circle $x = a\cos\theta$, $y = a\sin\theta$; is π if C is the circle $x - a = a\cos\theta$, $y = a\sin\theta$.

8. Find the area of the surface $z^2 = x^3$ included by the planes $x = 0$, $y = 0$, $x + y = 1$.

9. Find the area of the surface $z = x^{3/2} + y^{3/2}$ bounded by the planes $y = x,\ y = -x,\ x = a$.

10. Show that the area of the portion of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ within the cylinder $x^2 + y^2 - ax = 0$ is given by

$$4a\int_0^a\int_0^{\sqrt{ax-x^2}}\frac{dy\,dx}{(a^2 - x^2 - y^2)^{1/2}} = 4a\int_0^{\pi/2}\int_0^{a\cos\theta}\frac{r\,dr\,d\theta}{(a^2 - r^2)^{1/2}} = 4a^2\left(\frac{\pi}{2} - 1\right)$$

11. Find the area of the portion of the surface $x^2 + y^2 = z^2$ within the surface $x^2 + y^2 = 2x$.

12. Show that the portion of the surface $x^2 + y^2 + z^2 = 2z$ within the surface $z = x^2 + y^2$ has the area 2π.

13. Derive the formula for the area of a surface of revolution, § 144 (1), from the formula § 290 (1).

14. Fill in the details of the following proof that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$.

Set $\quad K = \int_0^a e^{-x^2}\,dx = \int_0^a e^{-y^2}\,dy \quad \therefore\ K^2 = \int_0^a\int_0^a e^{-(x^2+y^2)}\,dx\,dy$

Then $\qquad \int_0^{\pi/2}\int_0^a e^{-r^2}\,r\,dr\,d\theta < K^2 < \int_0^{\pi/2}\int_0^{a\sqrt{2}} e^{-r^2}\,r\,dr\,d\theta$

Hence $\lim\limits_{a\to\infty} K^2 = \pi/4$, and therefore $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$.

291. Surface integrals. Let $F(x, y, z)$ denote a function which is continuous on the piece of surface S described in § 290. Divide S into parts each of which can be inscribed in a sphere of radius ρ. Let δS denote any one of these parts and also its area, and let x, y, z denote any point of δS. When $\rho \to 0$, the sum $\Sigma F(x, y, z)\,\delta S$ will approach a limit. This limit is called the integral of $F(x, y, z)$ over S, and is denoted by $\iint_S F(x, y, z)\,dS$.

If the equation of S be $z = f(x, y)$, and D be the projection of S on the xy-plane, and γ the acute angle which the normal at a point of S makes with Oz, then

$$\iint_S F(x, y, z)\,dS = \iint_D F(x, y, z)\sec\gamma\,dx\,dy \qquad (1)$$

For, by § 282 (1), we can express δS in the form

$$\delta S = [f_x^2(\bar{x}, \bar{y}) + f_y^2(\bar{x}, \bar{y}) + 1]^{1/2}\,\delta D$$

where $(\bar{x}, \bar{y})$ denotes some point in δD; and, by §§ 282, Example,

$$\lim \Sigma F(x, y, z)[f_x^2(x, y) + f_y^2(\bar{x}, \bar{y}) + 1]^{1/2}\,\delta D = \iint_D F(x, y, z)\sec\gamma\,dx\,dy$$

292. Integrals over a designated side of a surface. 1. We may regard the piece of surface S of §§ 290, 291 as having two distinct sides.[1] At any point P of S there are two half normals PQ and PQ' one directed upward the other downward; we call them the normals to the upper and lower sides of S.

Let X, Y, Z denote the direction angles of normals to one side of S, and $R(x, y, z)$ a function which is continuous on S, and consider the integral

$$\iint_S R \cos Z \, dS \tag{1}$$

For the normals to the upper side of S, Z is the acute angle γ in § 291 (1), and if in that formula we replace F by $R \cos Z$, we find

$$\iint_S R \cos Z \, dS = \iint_D R \, dx \, dy \tag{2}$$

For normals to the lower side of S, Z is $\pi - \gamma$ and

$$\iint_S R \cos Z \, dS = - \iint_D R \, dx \, dy \tag{3}$$

In the first case the integral (1) is said to be taken over the upper side of S, in the second, over the lower side. In both cases it is customary to represent the integral (1) by the symbol $\iint_S R \, dx \, dy$. Hence, according as the integration is over the upper or the lower side of S, we have

$$\iint_S R \, dx \, dy = \iint_D R \, dx \, dy \quad \text{or} \quad - \iint_D R \, dx \, dy \tag{4}$$

The formulas (2) and (3) also hold good when $\cos Z$ becomes 0 on the contour of S. For if S' denote a variable interior portion of S, and D' its projection on the xy-plane, $\iint_{S'} R \cos Z \, dS = \pm \iint_{D'} R \, dx \, dy$, and this gives (2) or (3) when $S' \rightarrow S$.

Evidently $\iint_S R \cos Z \, dS$ is 0 when taken over a cylindrical surface parallel to Oz.

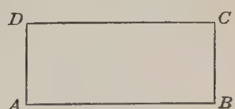

FIG. 133.

[1] By this it is meant that were S to be covered by a thin layer of some impenetrable substance, a point moving on one side of this layer could not pass to the other side without crossing the contour of S. One can form a surface which does not possess this property, and is therefore called one-sided, by twisting a rectangular piece of paper $ABCD$ so as to be able to fasten the edges BC and AD together with B on D and C on A.

2. Let T denote any two-sided surface which can be divided into pieces S of the type just considered, or into such pieces and portions of cylindrical surfaces parallel to Oz. We define

$$\iint_T R\,dx\,dy = \iint_T R \cos Z\,dS \tag{5}$$

over one side of T as the sum of the integrals over the corresponding sides of the parts S.

We have analogous definitions of the surface integrals

$$\begin{aligned} \iint_T P\,dy\,dz &= \iint_T P \cos X\,dS \\ \iint_T Q\,dz\,dx &= \iint_T Q \cos Y\,dS \end{aligned} \tag{6}$$

where P and Q are continuous functions of x, y, z on T; and with these three integrals, all taken over the same side of T, we form the general surface integral

$$\int_T P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy \tag{7}$$

Observe that in (5), (6), (7), dS is positive and the signs of $dx\,dy$, $dy\,dz$, and $dz\,dx$ are those of $\cos Z$, $\cos X$, $\cos Y$.

293. Green's formula for space. 1. Let V denote a closed region bounded below and above by pieces of surface S_1 and S_2 of the type described in § 290, and laterally by a cylindrical surface S_3 parallel to Oz. Call the entire surface S, and let X, Y, Z denote the direction angles of normals to its *outer* side.

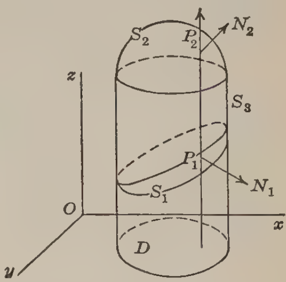

FIG. 134.

Let $R(x, y, z)$ be any function which, together with $\partial R/\partial z$, is continuous in V and on S. If z_2 and z_1 are the ordinates of the points, P_2 and P_1, where a parallel to Oz meets S_2 and S_1, we have

$$\iiint_V \frac{\partial R}{\partial z}\,dz\,dx\,dy = \iint_D R(x, y, z_2)\,dx\,dy - \iint_D R(x, y, z_1)\,dx\,dy \tag{1}$$

where D is the projection of S_2 and S_1 on the xy-plane. But since the outer normal to S is the upper normal along S_2 and the lower normal along S_1, we have, § 292 (2), (3),

$$\iint_D R(x, y, z_2)\, dx\, dy = \iint_{S_2} R \cos Z\, dS$$

$$\iint_D R(x, y, z_1)\, dx\, dy = -\iint_{S_1} R \cos Z\, dS$$

Hence the second member of (1) equals $\iint_{S_1} R \cos Z\, dS + \iint_{S_2} R \cos Z\, dS$, and, since $\iint_{S_3} R \cos Z\, dS = 0$, we have

$$\iiint_V \frac{\partial R}{\partial z}\, dx\, dy\, dz = \iint_S R \cos Z\, dS \tag{2}$$

2. The formula (2) is also true for any region V bounded by a closed surface S which can be divided into parts of the type just considered, — as can be shown by applying (2) to each of the parts and adding the resulting equations.

We establish analogous formulas for $\iiint_V (\partial P/\partial x)\, dx\, dy\, dz$, $\iiint_V (\partial Q/\partial y)\, dx\, dy\, dz$, and adding them and (2), obtain

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\, dy\, dz$$
$$= \iint_S (P \cos X + Q \cos Y + R \cos Z)\, dS \tag{3}$$

the surface integral being taken over the outer side of S. This is known as *Green's formula* for space.

By making the appropriate choices of P, Q, R in (3) we obtain the following formulas for the volume V:

$$V = \iint_S z\, dx\, dy = \iint_S x\, dy\, dz = \iint_S y\, dz\, dx \tag{4}$$

$$V = \tfrac{1}{3} \iint_S x\, dy\, dz + y\, dz\, dx + z\, dx\, dy \tag{5}$$

294. Stokes's theorem. Let S be a piece of surface of the type described in § 292 and let L be its contour; also let P, Q, R denote functions of x, y, z which, together with their

first partial derivatives, are continuous on S and L. We are to prove that

$$\int_L P\,dx + Q\,dy + R\,dz = \tag{1}$$
$$\iint_S \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy\right]$$

provided the following relation exists between the side of S and the sense along L for which the two integrals are taken: Take the x- and y-axes as in Fig. 135, so that the sense of rotation about Oz from Ox to Oy may be counter-clockwise, or positive. Suppose an observer to be standing on the side of S over which the integral $\iint_S$ is taken; then $\int_L$ is to be taken in the sense in which he must move along L to have S to his left.

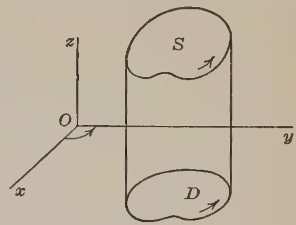

FIG. 135.

First consider the integral $\int_L P\,dx$.

Let D and C be the projections of S and L on the xy-plane and give the equation of S the form

$$z = f(x, y), \text{ also setting } p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

To fix the ideas, suppose $\iint_S$ to be taken over the upper side of S. The corresponding sense along L and C is counter clockwise as in the figure.

Finally, represent $P[x, y, f(x, y)]$ by $F(x, y)$.

We then have, by Green's theorem for the plane, § 287 (4),

$$\int_L P\,dx = \int_C F\,dx = -\iint_D \frac{\partial F}{\partial y}\,dy\,dx = -\iint_S \left[\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\,q\right] dx\,dy$$

But, if X, Y, Z be the direction angles of any normal to S, then

$$\cos X : \cos Y : \cos Z = -p : -q : 1, \quad \text{and} \quad dx\,dy = \cos Z\,dS$$

$$\therefore q\,dx\,dy = -\frac{\cos Y}{\cos Z}\cos Z\,dS = -\cos Y\,dS = -dz\,dx$$

Hence
$$\int_L P\,dx = \iint_S \frac{\partial P}{\partial z}\,dz\,dx - \frac{\partial P}{\partial y}\,dx\,dy$$

Similarly, from the projections of S and L on the yz- and zx-planes, we get

$$\int_L Q\,dy = \iint_S \frac{\partial Q}{\partial x}\,dx\,dy - \frac{\partial Q}{\partial z}\,dy\,dz$$

$$\int_L R\,dz = \iint_S \frac{\partial R}{\partial y}\,dy\,dz - \frac{\partial R}{\partial x}\,dz\,dx$$

Adding these three equations, we obtain (1). We suppose the projections of L on the xy-, yz-, zx-plane to be curves of the kind in § 286.

It is obvious from (1) that the value of the integral in the second member depends on P, Q, R and L only, not on S.

Suppose that throughout some region T, § 214, 2, we have

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \tag{2}$$

Then, by (1), the integral of

$$P\,dx + Q\,dy + R\,dz \tag{3}$$

along any closed curve L in T is 0. But from this it follows, as in § 288, that if $A(x_0, y_0, z_0)$ and $B(x, y, z)$ are points of T, the value of the integral of (3) along any path in T from A to B is independent of the path, and therefore, when A is fixed and B varies, is a one-valued function of the coordinates x, y, z of B. We call this function

$$F(x, y, z) = \int_{x_0, y_0, z_0}^{x, y, z} P\,dx + Q\,dy + R\,dz \tag{4}$$

It can be shown, as in § 288, that $\partial F/\partial x = P$, $\partial F/\partial y = Q$, $\partial F/\partial z = R$. Hence (3) is the differential of F and we have proved that (3) is an exact differential when the equations (2) are satisfied.

Furthermore if $u(x, y, z)$ be any known function whose differential is (3), then, as in § 288,

$$F(x, y, z) = u(x, y, z) - u(x_0, y_0, z_0) \tag{5}$$

We can find such a function $u(x, y, z)$ by the formula

$$u(x, y, z) = \int_{x_0}^{x} P(x, y_0, z_0)dx + \int_{y_0}^{y} Q(x, y, z_0)dy + \int_{z_0}^{z} R(x, y, z)dz \tag{6}$$

295. Work. Suppose that the point $P(x, y, z)$ moves from A to B under the action of a force F which is a continuous function of x, y, z and therefore of s, the length of the path AP. If θ denote the angle between the direction of F and the direction of the motion at any instant, then $F \cos \theta$ is the component of F in the direction of the motion at that instant. Therefore, by an obvious extension of the reasoning in § 147, we are led to define the work done by F as

$$W = \int_{AB} F \cos \theta \, ds \qquad (1)$$

The direction cosines of the motion at P are $\dfrac{dx}{ds}, \dfrac{dy}{ds}, \dfrac{dz}{ds}$; let those of F be l, m, n. Then

$$\cos \theta \, ds = \left[l \frac{dx}{ds} + m \frac{dy}{ds} + n \frac{dz}{ds}\right] ds = l\,dx + m\,dy + n\,dz$$

and therefore $\quad W = \int_{AB} Fl\,dx + Fm\,dy + Fn\,dz \qquad (2)$

Observe that Fl, Fm, Fn are the x-, y-, z-components of F at P. If Fl, Fm, Fn satisfy the conditions of § 294 (2), the value of W is independent of the path from A to B; $Fl\,dx + Fm\,dy + Fn\,dz$ is the differential of some function $\phi(x, y, z)$; and

$$W = \int_A^B d\phi = \phi(x, y, z)_B - \phi(x, y, z)_A \qquad (3)$$

In this case the force F is called a *conservative force.*

In general, $Fl\,dx + Fm\,dy + Fn\,dz$ is not an exact differential, and W depends on the path as well as on the positions of A and B.

EXERCISE LIX

1. A particle P moves from A to B under the action of a force directed toward O and of the magnitude k/OP^2; show that the work done is independent of the path.

2. Find the x-, y-, z-components of a force which would cause a particle P to move along the path $x = t$, $y = t^2$, $z = t^3$, t denoting time; also the work done between the times $t = 1$ and $t = 4$.

3. Show that $(yz\,dx - zx\,dy + xy\,dz)/y^2$ is an exact differential and find an integral by the formula § 294 (6).

4. Respecting the function u defined by § 294 (6), prove by aid of the theorem of § 283 that $\partial u/\partial x = P$, $\partial u/\partial y = Q$, $\partial u/\partial z = R$, if the conditions of § 294 (2) are satisfied.

5. Find the following surface integrals by means of the equivalent volume integrals:

1. $\iint zx\,dy\,dz + zy\,dz\,dx + z^2\,dx\,dy$ over the surface of one of the solids bounded by the sphere $x^2 + y^2 + z^2 = a^2$ and the planes $x = 0$, $y = 0$, $z = 0$.

2. $\iint xy\,dy\,dz + yz\,dz\,dx + xz\,dx\,dy$ over the surface of one of the solids bounded by the cylindrical surface $x^2 + y^2 = a^2$ and the planes $x = 0$, $y = 0$, $z = 0$, $z = b$.

6. Let T be a region bounded by two closed surfaces S_1 and S_2, S_2 lying entirely within S_1. Show that Green's formula, § 293 (3), holds good for T on the understanding that the integral $\iint_S$ is to be taken over the sides of S_1 and S_2 which are outward with respect to T.

7. Let T be a region, § 214, 2., in which P, Q, R, and their first partial derivatives are continuous. Show that the sufficient and necessary condition that

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy \tag{1}$$

be 0 for every closed surface S in T is that throughout T

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 \tag{2}$$

8. Let P be any point of a closed surface S, and (r, n) the angle which OP produced makes with the outward normal to S at P. Show that the volume V bounded by S is given by the formula

$$V = (1/3) \iint_S r \cos (r, n)\,dS.$$

9. Let S be a closed surface, M a given point, and (r, n) the angle which the outward normal at any point P of S makes with MP $(= r)$ produced; and let

$$I = \iint_S \frac{\cos (r, n)}{r^2}\,dS$$

Show that when S is a sphere and M its center, $I = 4\pi$; when S is a sphere and M a point on S, then $I = 2\pi$; when S is any closed surface, I is 4π, 2π, or 0 according as M is within, on, or outside of S.

10. Show that if S denote any piece of surface having a given contour L, as in § 294, the equation (2) of Ex. 7 is the condition that the value of (1) of Ex. 7 shall depend on L only, not on S.

To prove that (2) is the sufficient condition, show that if it be satisfied we can find functions A, B, C such that

$$P = \frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} \qquad Q = \frac{\partial A}{\partial z} - \frac{\partial C}{\partial x} \qquad R = \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \qquad (3)$$

and so reduce (1) to the line integral $\int_L A\,dx + B\,dy + C\,dz$ by Stokes's formula. The equations (3) imply (2); but if (2) be satisfied we can assign C, say, arbitrarily and then obtain A and B by simple integration.

To prove that (2) is the necessary condition, take any closed surface S' made up of parts S_1, S_2 of which L is the common contour. The sum of the integrals (1) taken over the outer and inner sides of the part S_1 is 0, and if the value of (1) is the same for every S having the contour L, it is the same for the inner side of S_1 as for the outer side of S_2; hence the value of (1) taken over the outer side of S' is 0, and this implies (2).

11. Show that if S be any piece of surface such as in Ex. 10, then

$$\iint_S x\,dy\,dz + y\,dz\,dx - 2\,z\,dx\,dy = \int_L yz\,dx + 3\,xz\,dy + 2\,xy\,dz$$

296. Parametric representation of surfaces. 1. A surface S may be represented by a set of equations of the form

$$x = \phi_1(u, v) \qquad y = \phi_2(u, v) \qquad z = \phi_3(u, v) \qquad (1)$$

where u, v denote independent variables or parameters. It is supposed that the functions ϕ_1, ϕ_2, ϕ_3 and their partial derivatives of the first order are continuous, and that none of the functional determinants

$$A = \frac{\partial(y, z)}{\partial(u, v)} \qquad B = \frac{\partial(z, x)}{\partial(u, v)} \qquad C = \frac{\partial(x, y)}{\partial(u, v)} \qquad (2)$$

vanishes identically. At an ordinary point P of S at least one of A, B, C is not 0. If $C \neq 0$, the first and second of the equations (1) can be solved at P for u, v in terms of x, y, § 230, and when the results are substituted in the third equation it takes the form $z = f(x, y)$. Points, if any, where A, B, C are all 0 are called *singular points* of S.

2. Any given pair of values u, v, as $u = a$, $v = b$, determines a single point P of S. Hence u, v are called the *curvilinear coordinates* of the points of S.

Any equation of the form $F(u, v) = 0$ represents a curve on S. In particular, $u = a$ represents a curve along which v alone varies and whose parametric equations are $x = \phi_1(a, v)$, $y = \phi_2(a, v)$, $z = \phi_3(a, v)$, the direction ratios of its tangent therefore being $\partial x/\partial v : \partial y/\partial v : \partial z/\partial v$. The like is true of the equation $v = b$.

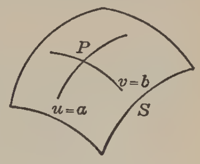

FIG. 136.

3. Let $P(x_1, y_1, z_1)$ be the point whose curvilinear coordinates are $u = a$, $v = b$, and let A_1, B_1, C_1 denote the values of A, B, C at P. Then if P is not a singular point of S, the equation of the tangent plane to S at P is

$$A_1(x - x_1) + B_1(y - y_1) + C_1(z - z_1) = 0 \tag{3}$$

For the normal at P is perpendicular to both of the curves $u = a$ and $v = b$ at P; hence if its direction cosines are l, m, n, we have at P

$$l\frac{\partial x}{\partial v} + m\frac{\partial y}{\partial v} + n\frac{\partial z}{\partial v} = 0, \quad l\frac{\partial x}{\partial u} + m\frac{\partial y}{\partial u} + n\frac{\partial z}{\partial u} = 0$$

and therefore
$$l : m : n = A_1 : B_1 : C_1 \tag{4}$$

4. If in the formula for the differential of curve arc, namely $ds^2 = dx^2 + dy^2 + dz^2$, we substitute the values of dx, dy, dz got from (1), we obtain for a curve on S,

$$ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2 \tag{5}$$

where
$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$F = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) + \left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial z}{\partial v}\right)$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

When the curves $u = \text{const.}$, $v = \text{const.}$ are orthogonal, then $F = 0$, § 212, and (5) becomes $ds^2 = E\,du^2 + G\,dv^2$.

5. Let S' denote a portion of S on which the normal is nowhere perpendicular to Oz. By § 290, we have

$$\text{area } S' = \iint_{D'} \frac{dx\,dy}{\cos\gamma} \tag{6}$$

where D' denotes the projection of S' on the xy-plane, and γ the acute angle which a normal to S' makes with Oz. We are to prove that the substitution $x = \phi_1(u, v)$, $y = \phi_2(u, v)$ of (1) transforms (6) into

$$\begin{aligned} \text{area } S' &= \textstyle\iint_D (A^2 + B^2 + C^2)^{1/2}\,du\,dv \\ &= \textstyle\iint_D (EG - F^2)^{1/2}\,du\,dv \end{aligned} \tag{7}$$

where D denotes the region (supposed to exist) in the uv-plane to which the region D' corresponds in the sense explained in § 289.

For by (3), and because $\cos\gamma$ is positive throughout D, we have

$$\cos\gamma = |C| \div (A^2 + B^2 + C^2)^{1/2}$$

Again C is the functional determinant J in the formula § 289 (3), and by hypothesis $\cos\gamma$ and therefore C does not vanish in D. Hence by this formula § 289 (3) we have immediately

$$\iint_{D'} \frac{dx\,dy}{\cos\gamma} = \iint_D (A^2 + B^2 + C^2)^{1/2}\,du\,dv$$

And, by using the expanded forms of A, B, C and the expressions for E, F, G given above, it is easily shown that

$$A^2 + B^2 + C^2 \equiv EG - F^2$$

When the curves $u = \text{const.}$, $v = \text{const.}$ are orthogonal, then

$$\text{area } S' = \textstyle\iint_D \sqrt{EG}\,du\,dv$$

EXERCISE LX

1. Let $P(x, y, z)$ be any point on the sphere $x^2 + y^2 + z^2 = a^2$. And let CPD be the great circle through P and the pole C. If $u = POC$ and $v = AOD$, show that

$$x = a \sin u \cos v \qquad y = a \sin u \sin v$$
$$z = a \cos u \tag{1}$$

The coordinate curves $u = c_1$ and $v = c_2$ through P, namely FPG and CPD, are orthogonal.

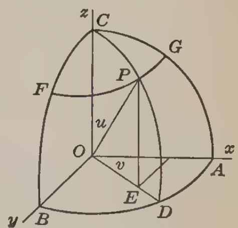

FIG. 137.

2. Show that for the sphere, Ex. 1., the formulas § 296 (5), (7) are

$$ds^2 = a^2(du^2 + \sin^2 u\, dv^2) \quad (2) \qquad dS = a^2 \sin u\, du\, dv \quad (3)$$

3. Show, by Ex. 2 (2), that the angle θ which a curve on the sphere through P makes with the curve $u = c$ through P is given by $\tan\theta = du/\sin u\, dv$; and therefore that the equation of a curve which cuts all the curves $u = c$ at the angle whose tangent is k is of the form

$$kv = \log\tan(u/2) + C \quad (\S\ 119, \text{Example})$$

4. Find the area of the portion of the sphere bounded by the curves $u = \pi/4$, $u = \pi/3$, $v = 0$, $v = \pi/6$; also that of the portion bounded by $v = 0$, $v = u$, $u = 0$, $u = \dfrac{\pi}{2}$.

5. Show that $x = u\cos v$, $y = u\sin v$, $z = ku$ are the parametric equations of a cone. Interpret u, v, k and find ds and dS for this cone.

6. Show that the equations $x = u$, $y = f(u)\cos v$, $z = f(u)\sin v$ represent a surface of revolution. Find the formula for its area between $x = a$ and $x = b$, and reduce this formula to the form § 144 (1).

7. Find the equation of the tangent plane at any point of the surface $x = u$, $y = v$, $z = 1/uv$.

8. Using the notation of § 296 (2), show that if $x = \phi_1(u, v)$, $y = \phi_2(u, v)$, $z = \phi_3(u, v)$ represent the surface $f(x, y, z) = 0$, then

$$f_x : f_y : f_z = A : B : C$$

9. Let θ be the angle between two curves on the surface $x = \phi_1$, $y = \phi_2$, $z = \phi_3$ which meet at the point (u, v) and there have the directions given by $du_1 : dv_1$ and $du_2 : dv_2$. Show that

$$\cos\theta = \frac{E\, du_1\, du_2 + F(du_1\, dv_2 + du_2\, dv_1) + G\, dv_1\, dv_2}{ds_1 \cdot ds_2}$$

10. Show that if ω be the angle between the coordinate curves $u = c_1$, $v = c_2$, then

$$\cos\omega = F/\sqrt{EG} \qquad \therefore \sin\omega = \sqrt{EG - F^2}/\sqrt{EG} \qquad \therefore dS = ds_u\, ds_v \sin\omega$$

which is the area of the parallelogram with sides of lengths ds_u, ds_v tangent to the curves $u = c_1$, $v = c_2$ at the point (c_1, c_2).

297. Transformation of surface integrals. Suppose that the formulas

$$x = \phi_1(x', y', z') \qquad y = \phi_2(x', y', z') \qquad z = \phi_3(x', y', z')$$

make the points of a given piece of surface S in the xyz-space correspond one to one to the points of a piece of surface S'

in the $x'y'z'$-space. And let $P(x, y, z)$ and its partial derivatives of the first order be continuous on S. What integral $\iint_{S'}$ corresponds to the integral $\iint_S P\,dx\,dy$?

We are to prove that

$$\iint_S P\,dx\,dy = \iint_{S'} P\left[\frac{\partial(x, y)}{\partial(y', z')}dy'\,dz' + \frac{\partial(x, y)}{\partial(z', x')}dz'\,dx' + \frac{\partial(x, y)}{\partial(x', y')}dx'\,dy'\right] \quad (1)$$

We shall assume that we may regard the coordinates x, y, z and x', y', z' of corresponding points of S and S' as functions of two independent variables or parameters u, v in a region D of the uv plane, one pair of these points of S and S' corresponding to each point of D and vice versa. Let

$$A = \frac{\partial(y, z)}{\partial(u, v)}, \quad B = \frac{\partial(z, x)}{\partial(u, v)}, \quad C = \frac{\partial(x, y)}{\partial(u, v)}, \quad K = (A^2 + B^2 + C^2)^{1/2}$$

$$A' = \frac{\partial(y', z')}{\partial(u, v)}, \quad B' = \frac{\partial(z', x')}{\partial(u, v)}, \quad C' = \frac{\partial(x', y')}{\partial(u, v)}, \quad K' = (A'^2 + B'^2 + C'^2)^{1/2}$$

Then if X, Y, Z denote the direction angles of the normals to one side of S, and X', Y', Z' those of the normals to one side of S', we have

$$\begin{aligned} A &= \pm K\cos X, & B &= \pm K\cos Y, & C &= \pm K\cos Z \\ A' &= \pm K'\cos X', & B' &= \pm K'\cos Y', & C' &= \pm K'\cos Z' \end{aligned} \quad (2)$$

the $\pm$ signs in each row being the same. By the formula § 285 (7), we have

$$C = \frac{\partial(x, y)}{\partial(y', z')}A' + \frac{\partial(x, y)}{\partial(z', x')}B' + \frac{\partial(x, y)}{\partial(x', y')}C'$$

Therefore, by (2),

$$K\cos Z = \pm K'\left[\frac{\partial(x, y)}{\partial(y', z')}\cos X' + \frac{\partial(x, y)}{\partial(z', x')}\cos Y' + \frac{\partial(x, y)}{\partial(x', y')}\cos Z'\right]$$

and we can make the $\pm$ sign $+$ by properly choosing the side of S' to which the normals are to be taken.

Multiply both members of this equation by $P\,du\,dv$ and integrate over D. Then since $K\,du\,dv = dS$ and $K'\,du\,dv = dS'$ by § 296 (7), we have

$$\iint_S P\cos Z\,dS$$
$$= \iint_{S'} P\left[\frac{\partial(x, y)}{\partial(y', z')}\cos X' + \frac{\partial(x, y)}{\partial(z', x')}\cos Y' + \frac{\partial(x, y)}{\partial(x', y')}\cos Z'\right]dS'$$

and, by § 292, this equation can be written in the form (1).

298. Transformation of triple integrals. Suppose that the formulas

$$x = \phi_1(x', y', z') \qquad y = \phi_2(x', y', z') \qquad z = \phi_3(x', y', z') \qquad (1)$$

make the points of the region V bounded by the closed surface S in the xyz-space correspond one to one to the points of the region V' bounded by the closed surface S' in the $x'y'z'$-space; also that in V' and on S', the functions ϕ_1, ϕ_2, ϕ_3 and their partial derivatives of the first and second orders are continuous, and the functional determinant

$$J = \frac{\partial(x', y', z')}{\partial(x, y, z)}$$

different from 0, and therefore of constant sign. Then, if $f(x, y, z)$ denote any function which is continuous in V and on S,

$$\int\int\int_V f(x, y, z)\, dx\, dy\, dz = \int\int\int_{V'} f(\phi_1, \phi_2, \phi_3)\, |\, J\, |\, dx'\, dy'\, dz' \qquad (2)$$

For by § 293 (4) we have $V = \int\int_S z\, dx\, dy$, and by § 297 (1) we can reduce $\int\int_S z\, dx\, dy$ to the form

$$\pm \int\int_{S'} z \left[\frac{\partial(x, y)}{\partial(y', z')}\, dy'\, dz' + \frac{\partial(x, y)}{\partial(z', x')}\, dz'\, dx' + \frac{\partial(x, y)}{\partial(x', y')}\, dx'\, dy'\right]$$

the two integrals being taken over the outer sides of S and S' respectively.

Let us transform the second integral into a triple integral by aid of Green's formula. If in the formula § 293 (3) we set

$$P = z\, \frac{\partial(x, y)}{\partial(y', z')} \qquad Q = z\, \frac{\partial(x, y)}{\partial(z', x')} \qquad R = z\, \frac{\partial(x, y)}{\partial(x', y')}$$

we obtain, on carrying out the indicated reckoning,

$$\frac{\partial P}{\partial x'} + \frac{\partial Q}{\partial y'} + \frac{\partial R}{\partial z'} = \frac{\partial(x, y, z)}{\partial(x', y', z')} = J$$

We therefore have

$$V = \pm \int\int\int_{V'} J\, dx'\, dy'\, dz' = \int\int\int_{V'} |\, J\, |\, dx'\, dy'\, dz' \qquad (3)$$

the third member following from the second because V is positive.

The formula (2) follows from (3) by an obvious extension of the argument given in § 289, 2.

Example. The formulas for transformation to polar coordinates in space are

$$x = r \sin \phi \cos \theta \qquad y = r \sin \phi \sin \theta \qquad z = r \cos \phi$$

where, in Fig. 137, $r = OP$, $\phi = POC$, and $\theta = AOD$. Show that

$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint_{V'} f(x, y, z) r^2 \sin \phi\, dr\, d\theta\, d\phi$$

XXVII. DOUBLE SERIES. IMPLICIT FUNCTIONS

299. Double series. It can be proved of any finite number of infinite series, Σa_n, Σb_n, $\cdots$, as was done for two such series in § 191, 1, that if they converge to the sums A, B, $\cdots$, the series got by adding their corresponding terms, namely $\Sigma(a_n + b_n + \cdots)$, will converge to the sum $A + B + \cdots$. The following theorem relates to the case in which the number of the given series is infinite, the resulting series $\Sigma(a_n + b_n + \cdots)$ then being called a *double series.*

Theorem. *Let $U_1 + U_2 + \cdots$ be a convergent series whose terms are themselves sums of absolutely convergent series:*

$$U_1 = u_1^{(1)} + u_2^{(1)} + \cdots,\ U_2 = u_1^{(2)} + u_2^{(2)} + \cdots, \cdots \qquad (1)$$

Also let

$$U'_1 = |\, u_1^{(1)} \,| + |\, u_2^{(1)} \,| + \cdots,\quad U'_2 = |\, u_1^{(2)} \,| + |\, u_2^{(2)} \,| + \cdots, \cdots$$

If the series $U'_1 + U'_2 + \cdots$ converges, then the several series obtained by adding the corresponding terms of the series (1), *namely*

$$u_1^{(1)} + u_1^{(2)} + u_1^{(3)} + \cdots,\quad u_2^{(1)} + u_2^{(2)} + u_2^{(3)} + \cdots, \cdots \qquad (2)$$

will also converge, and if their sums be V_1, V_2, $\cdots$, we shall have

$$U_1 + U_2 + U_3 + \cdots = V_1 + V_2 + V_3 + \cdots \qquad (3)$$

For let the remainders after n terms in the series (1) be $R_n^{(1)}$, $R_n^{(2)}$, $\cdots$; then

$$\begin{aligned} U_1 &= u_1^{(1)} + u_2^{(1)} + \cdots + u_n^{(1)} + R_n^{(1)} \\ U_2 &= u_1^{(2)} + u_2^{(2)} + \cdots + u_n^{(2)} + R_n^{(2)} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ U_k &= u_1^{(k)} + u_2^{(k)} + \cdots + u_n^{(k)} + R_n^{(k)} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Each of the column series, as $u_1^{(1)} + u_1^{(2)} + \cdots$, is convergent since each of its terms is numerically less than the corresponding term of the convergent positive series $U'_1 + U'_2 + \cdots$, §§ 168, 176. Let the sums of these series be $V_1, V_2, \cdots, V_n, R_n$.

If we add the corresponding terms of these $n+1$ column series, we obtain the original series $U_1 + U_2 + \cdots$. Therefore

$$U_1 + U_2 + \cdots + U_k + \cdots = V_1 + V_2 + \cdots + V_n + R_n$$

Hence the theorem will be proved if it can be shown that $\lim\limits_{n\to\infty} R_n = 0$.

Let the remainder after the kth term in $R_n^{(1)} + R_n^{(2)} + \cdots$ be $T_n^{(k)}$; then

$$R_n = R_n^{(1)} + R_n^{(2)} + \cdots + R_n^{(k)} + T_n^{(k)}$$

But if any positive number ϵ be assigned, we can first find a value k' of k such that $|T_n^{(k)}| < \epsilon/2$ when $k \geqq k'$, whatever the value of n may be, and then find a value n' of n such that $|R_n^{(1)}|, |R_n^{(2)}|, \cdots |R_n^{(k)}|$ are each $< \epsilon/2\,k$ when $n \geqq n'$. We shall then have $|R_n| < \epsilon$. And since ϵ may be taken as small as we please, $\lim\limits_{n\to\infty} R_n = 0$.

300. Operations with power series. The theorem of § 299 gives us a means of proving the theorems respecting power series stated in § 191.

1. Multiplication. If $f(x) = \Sigma a_n x^n$ and $\phi(x) = \Sigma b_n x^n$, for $|x| < l$, then also, for $|x| < l$, the product $f(x)\,\phi(x)$ is the sum of the series whose terms are

$$U_1 = f(x)b_0 \quad = a_0b_0 + a_1b_0x + a_2b_0x^2 + \cdots \qquad (1)$$

$$U_2 = f(x)b_1x \;= \;0 \;\;+ a_0b_1x + a_1b_1x^2 + \cdots \qquad (2)$$

$$U_3 = f(x)b_2x^2 = \;0 \;\;+ \;0 \;\;\; + a_0b_2x^2 + \cdots \qquad (3)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Since $\Sigma|a_nx^n|$ converges, the series obtained by replacing the terms of (1), (2), $\cdots$ by their absolute values converge to sums $U'_1, U'_2, \cdots$; and $U'_1 + U'_2 + \cdots$ converges since $\Sigma|b_nx^n|$ converges. Hence, § 299, the product $f(x)\,\phi(x)$ is also the sum of the series got by adding the terms of (1), (2), $\cdots$ by columns: that is

$$f(x)\,\phi(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots, \; |x| < l$$

2. Substitution. Let $z = \Sigma a_n y^n$, and let $y = \Sigma b_n x^n$, and suppose that the limits of convergence of the two series are l and r respectively.

If $|b_0| < l$, we can find[1] a positive number $r'(\leq r)$, such that $\Sigma|b_n x^n| < l$ when $|x| < r'$.

It being supposed that $|x| < r'$, replace $y, y^2, \cdots$ in $\Sigma a_n y^n$ by the series $\Sigma b_n x^n$ and the series derived from it by repeated applications of the rule in 1. The resulting double series satisfies the conditions of § 299, and if in it we group and add terms involving like powers of x, we obtain a power series in x which converges and represents z, at least when $|x| < r'$.

In case $l = \infty$ we have $\Sigma|b_n x^n| < l$ when $|x| < r$. Hence the substitution under consideration is valid for all finite values of x when $l = \infty$ and $r = \infty$.

3. Division. Consider $(a_0 + a_1 x + \cdots)/(b_0 + b_1 x + \cdots)$, a quotient of two power series which have limits of convergence > 0 and in which $b_0 \neq 0$. We are to prove that a power series $C_0 + C_1 x + \cdots$ exists which has a limit of convergence > 0 and which when convergent equals this quotient.

For a positive number c can be found such that when $|x| < c$ the series $a_0 + a_1 x + \cdots$ converges and also $|b_1 x| + |b_2 x^2| + \cdots < |b_0|$.

Let
$$y = b_1 x + b_2 x^2 + \cdots \tag{1}$$
and suppose $|x| < c$, and therefore $|y| < |b_0|$. We then have
$$\frac{1}{b_0 + b_1 x + \cdots} = \frac{1}{b_0 + y} = \frac{1}{b_0}\,\frac{1}{1 + y/b_0} = \frac{1}{b_0} - \frac{y}{(b_0)^2} + \frac{y^2}{(b_0)^3} - \cdots \tag{2}$$

Transform the series (2) into a power series in x by the substitution (1) and then multiply this x-series by the series $a_0 + a_1 x + \cdots$. The resulting x-series will converge and represent the given quotient, at least when $|x| < c$.

301. Implicit functions. 1. By § 225 an algebraic equation of the form
$$0 = ax + by + cx^2 + exy + fy^2 + \cdots \qquad b \neq 0 \tag{1}$$

[1] For let c be any positive number $< r$. Then since $\Sigma|b_n c^n|$ converges, each of its terms is less than some finite positive number $M < l$. Hence if $|x| < c$, we have
$$|b_n x^n| = |b_n c^n| \left|\frac{x}{c}\right|^n < M\left|\frac{x}{c}\right|^n$$
$$\therefore \qquad \Sigma|b_n x^n| < M\left[1 + \left|\frac{x}{c}\right| + \left|\frac{x}{c}\right|^2 + \cdots\right] = \frac{M}{1 - |x|/c}$$
and
$$\frac{M}{1 - |x|/c} < l \text{ when } |x| < c\left(1 - \frac{M}{l}\right) = r'.$$

has a solution $y = \phi(x)$ at the origin. We are to prove that $\phi(x)$ can be expressed as a power series in x with a limit of convergence $l > 0$.

By transposing the y term we can reduce (1) to the form:

$$y = (a_{10}x + a_{20}x^2 + \cdots) + y(a_{11}x + a_{21}x^2 + \cdots)$$
$$+ y^2(a_{02} + a_{12}x + a_{22}x^2 + \cdots) + \cdots \quad (2)$$

In (2) set
$$y = C_1x + C_2x^2 + C_3x^3 + \cdots \quad (3)$$

and, after reducing the second member to a power series in x, equate coefficients of like powers of x in the two members. We thus get

$$C_1 = a_{10}, \quad C_2 = a_{20} + C_1a_{11} + C_1^2a_{02}, \cdots$$

Each of these equations after the first expresses one of the C's as a polynomial in certain of the a_{ik}'s and the preceding C's. Therefore when solved they give expressions for the C's as polynomials in the a_{ik}'s. Thus

$$C_1 = a_{10}, \; C_2 = a_{20} + a_{10}a_{11} + a_{10}^2a_{02}, \cdots \quad (4)$$

The series (3) with the coefficients (4) satisfies (2) formally. It is an *actual* solution of (2) if, as has been assumed in deriving it, it has a limit of convergence $l > 0$. To prove that it has, let λ denote the value of the greatest $|a_{ik}|$, and form the equation

$$y = \lambda(x + x^2 + \cdots) + \lambda y(x + x^2 + \cdots)$$
$$+ \lambda y^2(1 + x + x^2 + \cdots) + \cdots \quad (5)$$

the right member being an infinite series in y with coefficients which are infinite series in x of the forms indicated.

Proceeding as in the case of (2), we can find a series

$$y = C'_1x + C'_2x^2 + C'_3x^3 + \cdots \quad (6)$$

which satisfies (5) formally. Moreover, since $\lambda \geqq |a_{ik}|$, we shall have, by (4), $C'_1 \geqq |C_1|$, $C'_2 \geqq |C_2|$, and so on. Hence if (6) has a limit of convergence $l > 0$, the same must be true of (3).

But if we add $\lambda + \lambda y$ to both members of (5) and suppose $|x|, |y| < 1$, we have

$$(\lambda + 1)y + \lambda = \lambda/(1 - x)(1 - y)$$

$$\therefore \qquad (\lambda + 1)y^2 - y + \lambda x/(1 - x) = 0 \tag{7}$$

Solving (7) for the root y which vanishes with x, we get

$$y = \{1 - [1 - (2\lambda + 1)^2 x]^{1/2}(1 - x)^{-1/2}\} \div 2(\lambda + 1) \tag{8}$$

By the binomial theorem and § 300, 1., we can expand the right member of (8) in a power series in x which converges when $(2\lambda + 1)^2 |x| < 1$, or

$$|x| < 1/(2\lambda + 1)^2 \tag{9}$$

And (9) and (8) imply that $|x|, |y| < 1$. The series thus obtained is identical with (6), by § 186. Hence (6) and therefore (3) has a limit of convergence $l > 0$, as was to be proved.

Example. Solve $y^2 - y + x = 0$ for y in terms of x at 0.

In $y = x + y^2$ substitute $\quad y = C_1x + C_2x^2 + C_3x^3 + \cdots$

We get $\; C_1x + C_2x^2 + C_3x^3 + \cdots = x + x^2(C_1 + C_2x + C_3x^2 + \cdots)^2$

Hence

$C_1 = 1, \quad C_2 = C_1^2 = 1, \quad C_3 = 2\,C_1C_2 = 2, \quad C_4 = C_2^2 + 2\,C_1C_3 = 5, \cdots$

Therefore $\quad y = x + x^2 + 2\,x^3 + 5\,x^4 + \cdots, \quad |x| < 1/(2 + 1)^2.$

302. Method of successive approximations. We may write the equation § 301 (2) in the form

$$y = cx + u_2 + u_3 + u_4 + \cdots \tag{1}$$

where $c = a_{10}$, and $u_2, u_3, \cdots$ denote the groups of terms of the second, third, $\cdots$ degrees in x, y.

It has been proved that the equation (1) has a solution of the form :

$$y = cx + C_2x^2 + C_3x^3 + C_4x^4 + \cdots \tag{2}$$

We call $y = cx$, $y = cx + C_2x^2$, $\cdots$ the *first, second,* $\cdots$ *approximations* to (2).

Since (2) satisfies (1) identically, the series $cx + C_2x^2 + C_3x^3 + \cdots$ must be identical with the x-series S got by substituting (2) in the second member of (1), expanding, and collecting terms in like powers of x.

But the only x^2 terms in S are those got by setting $y = cx$ in u_2. Hence we can get the second approximation $y = cx + C_2x^2$ directly from (1) by discarding u_3, u_4, $\cdots$ and substituting the first approximation $y = cx$ in u_2.

In like manner, the second approximation having been found, we can get the third by the process : In (1) discard u_4, $\cdots$, substitute the first approximation in u_3, and the second in u_2, ignoring the x^4 term in the result. And so on.

This is the most expeditious method of finding the solution (2).

EXAMPLE 1. Solve $y = 2x + y^2 + y^3$ for y in terms of x at O.

1st approx'n. $y = 2x$

2nd approx'n. $y = 2x + (2x)^2 = 2x + 4x^2$

3rd approx'n. $y = 2x + (2x + 4x^2)^2 + (2x)^3 = 2x + 4x^2 + 24x^3$

4th approx'n. $y = 2x + (2x + 4x^2 + 24x^3)^2 + (2x + 4x^2)^3$
$= 2x + 4x^2 + 24x^3 + 160x^4$

EXAMPLE 2. Find the shape of the graph K of $y = x + 2y^2 + x^2$ near O.

The first approximation $y = x$ represents the tangent OT.

The second approximation is $y = x + 3x^2$. When $x = OD$ is very small the term $3x^2$ is approximately equal to QP. Therefore, since $3x^2$ is always $+$, K lies above OT near O.

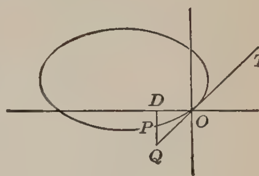

FIG. 138.

EXAMPLE 3. Find the fourth approximation to the solution of the equation $0 = x - 2y + xy + y^3 + x^4$ for y at O.

EXAMPLE 4. Find the shape of the graph near O of the equation $x + 2y - xy + y^2 - x^3 = 0$.

303. Newton's Diagram. Let $f(x, y) = 0$ denote any irreducible algebraic equation in x, y which lacks the constant term. Let us inquire whether in every case, that is, even when the y term is missing, $f(x, y) = 0$ has a solution of the

form $y = Cx^\mu + \cdots$, the series being one in increasing positive powers of x, integral or fractional.

Evidently if $y = Cx^\mu + \cdots$ is to satisfy $f(x, y) = 0$ identically, it is necessary first of all that μ be such as to make the degrees of two or more of the terms of $f(x, Cx^\mu)$ equal and less than the degrees of the remaining terms, and then that C be such as to make the sum of these terms of lowest degree 0.

It is therefore required to find all substitutions $y = x^\mu$ which will make the degrees in x of two or more of the terms of $f(x, y)$ equal and less than the degrees of the remaining terms. We proceed as follows:

Apart from numerical coefficients, the terms of $f(x, y)$ are of the type $x^k y^l$, and when $y = x^\mu$ the degree of $x^k y^l$ in x is $k + l\mu$.

1. Take rectangular axes OX, OY, and for any term $x^k y^l$ plot the point P whose X, Y coordinates are the exponents k, l; we call $P(k, l)$ the *degree point* of the term $x^k y^l$. Next, supposing $y = x^\mu$ and that μ is known, take $OM = \mu$ and $OI = 1$, and through P take AB parallel to MI.

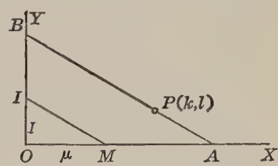

FIG. 139.

The equation of AB is

$$(X - k)/\mu + (Y - l)/1 = 0$$

Hence the X-intercept OA is $k + l\mu$, *the degree of* $x^k y^l$ *in* x, *and the Y-intercept OB is* $k/\mu + l$, *the degree of* $x^k y^l$ *in* y.

2. Plot the degree points $P, Q, \cdots$ of all the terms of $f(x, y)$. Then find every line L which will pass through two or more of these points and have the rest to the side remote from O. Let $x^k y^l$ and $x^{k'} y^{l'}$ be two of the terms whose points are on such a line L, and determine μ so that their degrees in x when $y = x^\mu$, are equal: which makes $k + l\mu = k' + l'\mu$, and therefore $\mu = (k' - k)/(l - l')$. Then, as follows from 1., all the terms whose points are on L will be of the same degree $k + l\mu$ in x, and all the other terms will be of a higher

degree, their degrees increasing with the distances of their points from L.

We call the groups of terms determined by the lines L the *possible groups of terms of lowest order* in $f(x, y) = 0$. By equating each such group to 0, discarding factors in x or y only, and then solving for y, we find the initial term, or first approximation, of every solution $y = Cx^{\mu} + \cdots$ that $f(x, y) = 0$ can have at O, μ being a positive rational number, integral or fractional.

Example. For $f(x, y) = xy^3 + 3x^2y^2 + 2x^3y - 3x^5 - y^6 + x^7 = 0$, the degree points are those indicated in Fig. 140, and the lines L are BC, CDE, EF.

The corresponding possible groups of terms of lowest order are

$$-3x^5 + 2x^3y$$
$$2x^3y + 3x^2y^2 + xy^3$$
$$xy^3 - y^6$$

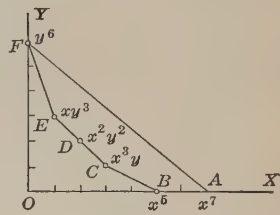

Fig. 140.

Equating each group to 0 and solving for y, we obtain

$$y = (3/2)x^2;\ y = -x,\ y = -2x;\ y = x^{1/3},\ y = \alpha x^{1/3},\ y = \alpha^2 x^{1/3},$$

where $$\alpha = (-1 + i\sqrt{3})/2$$

These are the initial terms of the possible solutions $y = Cx^{\mu} + \cdots$ of $f(x, y) = 0$ at O.

304. Solutions of algebraic equations at the origin. 1. Taking any one of the first approximations just found, $y = Cx^{\mu} (\mu = p/q)$, substitute $y = x^{\mu}(C + v)$ in the given equation $f(x, y) = 0$ of § 303. We obtain an algebraic equation in v and $x^{1/q}$, call it $\phi(v, x^{1/q}) = 0$, which lacks the constant term but ordinarily has the v term, and which may therefore be solved for v in powers of $x^{1/q}$ by the method of § 301. When this value of v is substituted in $y = x^{\mu}(C + v)$, we obtain a solution $y = \phi(x)$ of $f(x, y) = 0$ in the form of a series in ascending powers of $x^{1/q}$, with a limit of convergence $l > 0$.

In case the v term is missing in $\phi(v, x^{1/q}) = 0$, we may deal with this equation as we have just dealt with $f(x, y) = 0$.

It can be shown that it is always possible by a limited number of steps of this kind to obtain equations in which the first power of the dependent variable is present and which therefore will lead back to solutions of $f(x, y) = 0$.

If in $f(x, y)$ the term of lowest degree among those that involve y only is the y^q term, the number of the solutions of $f(x, y) = 0$ for y at O is q. See § 303, Example.

2. The most expeditious method of finding the solutions whose existence has thus been proved is that of successive approximations described in § 302.

Let $y^q - Cx^p$ occur once or $r > 1$ times as a factor of one of the groups of terms of lowest order in $f(x, y) = 0$, so that $y = C^{1/q}x^{p/q}$ is the first approximation to one or more of the required solutions. The second, third, $\cdots$ approximations may be obtained by first expressing $f(x, y) = 0$ in the form

$$(y^q - Cx^p)^r = u_1 + u_2 + u_3 + \cdots \tag{1}$$

where $u_1, u_2, \cdots$ denote terms or groups of terms arranged in the ascending order of their degrees in x when $y = C^{1/q}x^{p/q}$, and then proceeding as in § 302.

EXAMPLE 1. In $y^2 - 4x^2 - x^3 - y^4 = 0$ (a) the group of lowest order is $y^2 - 4x^2 = (y - 2x)(y + 2x)$.

To find the solution whose first approximation is $y = 2x$, write (a) in the form

$$y - 2x = \frac{x^3}{y + 2x} + \frac{y^4}{y + 2x} \tag{b}$$

Substituting $y = 2x$ in $x^3/(y + 2x)$, we get the second approximation

$$y = 2x + \frac{x^3}{4x} = 2x + \frac{x^2}{4} \tag{c}$$

Substituting (c) in $x^3/(y + 2x)$, and $y = 2x$ in $y^4/(y + 2x)$, we get the third approximation

$$\begin{aligned} y = 2x + \frac{x^3}{4x + x^2/4} + \frac{16x^4}{4x} &= 2x + \frac{x^2}{4}\left(1 - \frac{x}{16} + \cdots\right) + 4x^3 \\ &= 2x + \frac{x^2}{4} + \frac{255}{64}x^3 \end{aligned} \tag{d}$$

If $y^q - Cx^p$ is a factor of u_1, we retain u_2 in computing the second approximation.

Thus in the case of $(y - x)^2 = 2(y - x)x^2 + 3\,y^4$, the first approximation is $y = x$. But $y - x$ is a factor of u_1. We therefore substitute $y = x$ in $u_2 = 3\,y^4$ and then solve the equation as a quadratic in $y - x$, thus obtaining the second approximations $y = x + 3\,x^2$, $y = x - x^2$.

EXAMPLE 3. Find the fourth approximation to one of the solutions of $(y - x)^2 - y^3 = 0$ at O.

EXAMPLE 4. Find the second approximation to each of the solutions of $y^4 - xy^2 + 4\,x^3 = 0$ at O.

305. Singular points of algebraic curves. 1. Such of the solutions of $f(x, y) = 0$ considered in § 304 as have real coefficients represent curves which are called the "branches" of the graph K of $f(x, y) = 0$ at and near O. These branches differ very little near O from the graphs of the approximations to the solutions.

The first approximations are given by irreducible equations of one of the two types $y^q - Cx^p = 0 (q \neq p)$, and $y - Cx = 0$. To each equation $y^q - Cx^p = 0$ corresponds a branch of K which near O almost coincides with the curve $y^q - Cx^p = 0$ and, like it, touches Ox or Oy at O according as $q \lessgtr p$. To each equation $y - Cx = 0$ corresponds a branch of K which touches the line $y - Cx = 0$. Near O this branch almost coincides with the graph of the second approximation to the solution of $f(x, y) = 0$ corresponding to $y - Cx = 0$.

EXAMPLE 1. Find the graph K of $x^4 + y^4 - xy(x + y) = 0$ (*a*)

The possible groups of terms of lowest order are

$$x^4 - x^2y, \qquad y^4 - xy^2, \qquad - xy(x + y)$$

Setting $x^4 - x^2y = 0$ gives $y = x^2$

Hence there is a branch AOB which close to O nearly coincides with the parabola $y = x^2$.

Setting $y^4 - xy^2 = 0$ gives $y^2 = x$. Hence the branch DOE.

Setting $- xy(x + y) = 0$ gives $y = - x$. The second approximation of the corresponding solution of (*a*) is

$$y + x = \left[\frac{x^4 + y^4}{xy}\right]_{y=-x} \quad \text{or} \quad y = - x - \frac{x^2}{2}$$

which represents a parabola touching the line $y = -x$ at O and lying below it. Hence FOG.

The substitution $y = tx$, § 93, in (*a*) gives $\quad x = t(t+1)/(t^4+1)$

$$y = t^2(t+1)/(t^4+1)$$

and these equations show that as t increases from $-\infty$ to ∞, $P(t)$ traces successively $OEFO$, $OGAO$, $OBDO$.

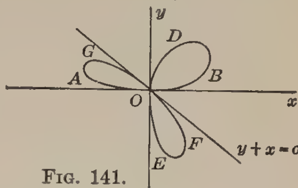

FIG. 141.

EXAMPLE 2. Find the graph of $x^4 + 2y^4 + (x-y)^2(2x+y) = 0$

2. A point of the graph of an equation $F(x, y) = 0$ at which both F_x and F_y are 0 is called a singular point, § 42, 4. Hence in the case of an equation $f(x, y) = 0$ which lacks the constant term, O is a singular point when $f(x, y)$ also lacks the x term and the y term (Exs. 1 and 2). In general the three equations $F = 0$, $F_x = 0$, $F_y = 0$ have no common solution, and therefore the curve $F = 0$ has no singular point. But if such a point (a, b) exists (found by solving $F_x = 0$, $F_y = 0$), its character can be determined by transforming $F(x, y) = 0$ into an equation $f(x', y') = 0$ which lacks the constant term — the method is explained in the next section — and then proceding as in 1.

306. Continuity of algebraic functions. 1. Since all of the solutions at O of the equation $f(x, y) = 0$ of § 304 are of the type $y = x^\mu(C + v)$, where μ is positive and $\lim_{x \to 0} v = 0$, all of them are continuous functions of x at O.

2. If (a, b) be any point at which a given irreducible algebraic equation $F(x, y) = 0$ is satisfied, and we make the substitution $x = a + x'$, $y = b + y'$, and then apply Taylor's theorem, we transform $F(x, y) = 0$ into an equation $f(x', y') = 0$ which is satisfied at $x', y' = 0, 0$. We can get the solutions $y' = \phi(x')$ of $f(x', y') = 0$ at $x', y' = 0, 0$ by § 304. If in each of them we substitute $x' = x - a$, $y' = y - b$, we get the solutions $y - b = \phi(x - a)$ of $F(x, y) = 0$ at (a, b). Hence *all of these solutions are continuous functions of x at* (a, b).

307. Infinite values of x and y. 1. An irreducible algebraic equation $f(x, y) = 0$ being given, let us inquire how y varies when $x \to \infty$.

In Newton's diagram, with the constant term of $f(x, y)$, if any, represented by O, each line L, with a positive X-intercept, which passes through two or more of the degree points of $f(x, y)$ and has the rest to the *origin* side, determines a possible group of terms of *highest* order with respect to x. Equate this group to 0, discard factors in x or y only, and solve for y in terms of x. We thus obtain one or more equations of the type $y = Cx^\mu$, where μ is rational and $\gtreqless 0$ according as the Y-intercept of L is $+$, ∞, or $-$.

Corresponding to each such equation $y = Cx^\mu$, substitute $y = x^\mu(C + v)$ in $f(x, y) = 0$, and then divide throughout by the highest power of x that appears in any of the terms. We thus obtain an equation in v and negative powers of x which lacks the constant term but ordinarily has the v term. By the reasoning of § 304, this equation has one or more solutions of the form

$$v = a/x^\alpha + b/x^\beta + \cdots \quad (\alpha > 0, \beta > \alpha, \cdots) \tag{1}$$

there being a number $l > 0$ such that the series on the right converges when $1/|x| < l$, and therefore when $|x| > l$.

Substituting (1) in $y = x^\mu(C + v)$, we get

$$y = x^\mu[C + a/x^\alpha + b/x^\beta + \cdots] \quad |x| > l \tag{2}$$

When $\mu > 0$, *then* $\lim\limits_{x \to \infty} y/x^\mu = C$, *and we say that* y *becomes infinite to the order* μ *with respect to* x.

When $\mu = 0$, $\lim\limits_{x \to \infty} y = C$; *when* $\mu < 0$, $\lim\limits_{x \to \infty} y = 0$.

We may obtain (2) by the method of successive approximations. We first arrange the terms of $f(x, y) = 0$ in the descending order of their degrees with respect to x when $y = Cx^\mu$, and then proceed as in § 304, 2.

2. In like manner, each line L with a positive Y-intercept determines a possible group of terms of highest order with respect to y and leads to one or more solutions of $f(x, y) = 0$ of the type

$$x = y^{\mu}[C' + a'/y^{\alpha'} + b'/y^{\beta'} + \cdots] \quad |y| > l' \tag{3}$$

EXAMPLE 1. The possible groups of terms of highest order with respect to x in

$y^3 - x^2y + x = 0$ (a) are $y^3 - x^2y$ (b) and $-x^2y + x$ (c)

$y^3 - x^2y = 0$ gives $y = x$, $y = -x$; $-x^2y + x = 0$ gives $y = 1/x$.

Substituting $y = x(1 + v)$ in (a), we get

$$x^3(1 + v)^3 - x^3(1 + v) + x = 0$$
$$\therefore 2v + 1/x^2 + 3v^2 + v^3 = 0 \tag{d}$$

Solving (d) for v,

$$v = -1/2\,x^2 - 3/8\,x^4 \cdots$$
$$\therefore y = x(1 + v) = x - 1/2\,x - 3/8\,x^3 \cdots \tag{e}$$

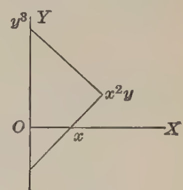

FIG. 142.

Similarly, corresponding to $y = -x$ and $y = 1/x$, we find

$$y = -x - 1/2\,x + 3/8\,x^3 \cdots \quad \text{and} \quad y = 1/x + 1/x^5 + 3/x^9 \cdots$$

EXAMPLE 2. For the equation $y^2x - 3xy - y^2 + 2x = 0$ (a), the line AC determines a group of highest order with respect to x, namely $y^2x - 3xy + 2x$ (b), and BC a group of highest order with respect to y, namely $y^2x - y^2$ (c).

Setting (b) $= 0$ gives $y = 1$, $y = 2$. To find the solution corresponding to $y = 1$, we write (a) in the form

$$y - 1 = \frac{y^2}{x(y - 2)} \tag{d}$$

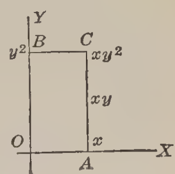

FIG. 143.

Hence the 2nd approximation, given by $y - 1 = \dfrac{1}{x(-1)}$, is $y = 1 - \dfrac{1}{x}$

and the 3rd approximation, given by $y - 1 = \dfrac{(1 - 1/x)^2}{x(-1 - 1/x)}$, is

$$y = 1 - \frac{1}{x} + \frac{3}{x^2}$$

Similarly corresponding to $y = 2$, we find $y = 2 + 4/x + 16/x^3 + \cdots$

Equating (c) to 0 gives $x = 1$. Hence we express (a) in the form $x - 1 = 3x/y - 2x/y^2$ and by successive approximations find

$$x = 1 + 3/y + 7/y^2 + \cdots$$

3. When $f(x, y)$ has a constant term, so that O is one of the degree points, and P is another degree point (not on OX or OY) such that all the degree points not on the line OP lie to one side of it; then if these remaining points are above OP, OP is a line of terms of highest (zero) degree which leads to solutions of the type $y = Cx^\mu + \cdots$, if below, to solutions of the type $x = Cy^\nu + \cdots$.

Thus for the equation $xy - y - 1 = 0$ the line joining the degree points of xy and -1 gives the solution $y = 1/x + 1/x^2 + \cdots$; the other solution $x = 1 + 1/y$ being given by the line joining the degree points of xy and $-y$.

308. Infinite branches of algebraic curves. Each of the solutions § 307 (2) which has real coefficients represents, for $x > l$, or $x < -l$, or both, an infinite branch of the curve $f(x, y) = 0$. By § 51, 2., a branch represented by (2) when $\mu = 1$ and $\alpha = 1$, that is, by $y = Cx + a + b/x^{\beta-1} + \cdots$, has the asymptote $y = Cx + a$. It is therefore called a *hyperbolic branch.* The branch is also hyperbolic when $\mu = 1$ and $\alpha > 1$, when $\mu = 0$, and when $\mu < 0$, the corresponding asymptotes being $y = Cx$, $y = C$, and $y = 0$.

In the remaining cases a branch represented by (2) has no asymptote, but it approaches parallelism with some given line, as does the half parabola $y = x^{1/2}$ with the x-axis; it is therefore called *a parabolic branch.*

The like is to be said regarding the infinite branches represented by § 307 (3) when real. The solutions considered in § 307, 3. all represent hyperbolic branches.

A knowledge of the infinite branches of the curve, when coupled with information which may be obtained directly from its equation $f(x, y) = 0$ — as the points where the curve cuts Ox and Oy, the finite points where it cuts its asymptotes, and its singular points, if any — often enable one to determine the general course of the curve without serious difficulty. Observe in particular that an asymptote, since it plays the

role of a tangent, cannot meet a curve whose equation is of degree n in more than $n - 2$ finite points.

EXAMPLE 1. Find the asymptotes and graph of

$$x^2y + xy^2 - x^2 - y^2 = 0 \qquad (a)$$

The possible groups of terms of highest order are

$$xy(y + x) \qquad x^2(y - 1) \qquad y^2(x - 1)$$

Solving (a), as in § 307, Ex. 2, for $y + x$, $y - 1$, $x - 1$, we get

$$y = -x - 2 - \frac{4}{x^2} \cdots \qquad (b)$$

$$y = 1 - \frac{1}{x} \cdots \qquad (c)$$

$$x = 1 - \frac{1}{y} \cdots \qquad (d)$$

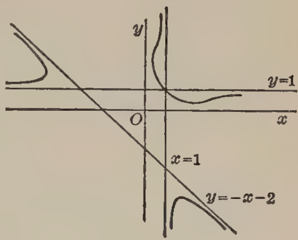

FIG. 144.

Hence the asymptotes are $y = -x - 2$, $y = 1$, $x = 1$. Moreover the term $-4/x^2$ in (b) shows that the infinite branches represented by (b) toward the $x = \infty$ and $x = -\infty$ "ends" of $y = -x - 2$ both lie below this asymptote. Similarly, the terms $-1/x$ in (c) and $-1/y$ in (d) show that the end parts of the corresponding infinite branches are to the sides of the asymptotes indicated in Fig. 144. It will be found that $y = 1$ and $x = 1$ both meet the curve at $(1, 1)$, but $y = -x - 2$ at no finite point. The group of terms of lowest order in (a) is $x^2 + y^2$; hence O is a conjugate point.

EXAMPLE 2. In $x^2y^2 - x^3 - y^3 = 0$ (a) the possible groups of terms of highest order are $x^2y^2 - x^3$ and $x^2y^2 - y^3$. When equated to 0, they give $y^2 - x = 0$ (b) and $x^2 - y = 0$ (c)

Hence there are no asymptotes. Parts of the infinite branches of (a) tend two to parallelism with Ox, like those of (b), and two to parallelism with Oy, like those of (c). They are parabolic branches.

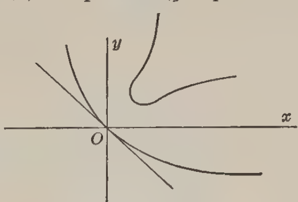

FIG. 145.

The curve is symmetric with respect to the line $y = x$, which cuts it at (1,1).

EXAMPLE 3. Find the graph of $(y^2 - x^2)x - y(y - 3x) = 0$.

The asymptotes are $y = x - 1$, $y = -x - 2$, $x = 1$

The tangents at O are $y = 3x$, $y = 0$

The substitution $y = tx$ gives

$$x = \frac{t(t-3)}{t^2-1} \quad y = \frac{t^2(t-3)}{t^2-1}$$

These equations show that, as t increases from $-\infty$ to ∞, $P(t)$ traces successively the arcs marked (1) ⋯ (5) in Fig. 146.

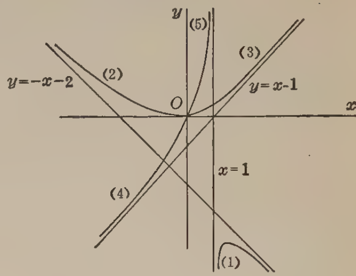

FIG. 146.

EXERCISE LXI

1. Let $f(x, y) = 0$ denote an irreducible algebraic equation of the nth degree, u_n its group of terms of the nth degree, and C its graph; prove the following, by aid of §§ 305, 307 :

1. If u_n has no real factor, then C has no infinite branch.
2. The asymptotes of C, if any, are parallel to the real lines through O represented by $u_n = 0$.
3. An asymptote l cannot meet C in more than $n - 2$ finite points.
4. Ordinarily there are portions of C near both "ends" of an asymptote l and on opposite sides of l. If they are on the same side, or at the same end, l meets C in less than $n - 2$ finite points.
5. In the case $n = 3$, an asymptote l ordinarily meets C in one finite point.
6. If u_k $(k > 0)$ denote the group of terms of lowest degree in $f(x, y)$, then C passes through O, and the tangents to C at O are the real lines represented by $u_k = 0$.

2. Trace the following curves :

1. $y^2 + 2xy - 3x^2 - 2y = 0$
2. $y^2 - 4x^2 + x - y + 3 = 0$
3. $x^2y + y^2x + y - 2x = 0$
4. $x^2y - xy^2 + y^2 - 4x^2 = 0$
5. $x^3 - y^3 + y^2 = 0$
6. $(y^2 - x^2)(y - 2x) - 2x^2 = 0$
7. $x^4 - y^4 + x^2 = 0$
8. $xy^2 + x - y = 0$
9. $x^3 - x^2y - 2xy^2 - 5xy + 2y^2 = 0$
10. $y^4 - xy^2 + 3x^2y - 2x^3 = 0$
11. $y^2x + x^2 - y = 0$
12. $(x - 1)y^2 + y - 2x = 0$
13. $(x - 1)^2y^2 = x - y$
14. $(y - x)^2(y + x) + y^2 - 3x^2 = 0$
15. $(y - 2x)^2y - x = 0$
16. $(x^2 - 1)y^2 = y - x$
17. $x^2y^2 + x^3 - y^3 + x^2 - y^2 = 0$
18. $(y - x)^2(y + x) + 2(y - x)x - y = 0$

309. Uniform convergence. Consider a series Σu_n whose terms are functions of x, and which converges for all values of x in a certain interval (a, b). Let S denote the sum of Σu_n for values of x in (a, b), also S_n the sum of n terms and R_n the remainder after n terms. Then S is a function of x in (a, b), and $S = S_n + R_n$.

Suppose that, any positive number ϵ having been assigned, it is always possible to find a value n' of n, which is independent of x, and is such that when $n \geqq n'$ then $|R_n| < \epsilon$ for *all values of* x in (a, b). We then say that the series is *uniformly convergent* in (a, b).

The proof in § 181 shows that any power series $\Sigma a_n x^n$ is uniformly convergent in any interval (a, b) which lies *within* its interval of convergence $(-l, l)$, and by an obvious extension of the reasoning of that section, it can be proved that

If for all values of x in (a, b) the terms of Σu_n are continuous, and are numerically less than the corresponding terms of a convergent positive series ΣM_n, then Σu_n is uniformly convergent, and its sum S is a continuous function of x in (a, b).

As an example of non-uniform convergence, consider the series

$$x + x(1-x) + x(1-x)^2 + \cdots$$

It converges in the interval $(0, 1)$, but not uniformly. For it will be found, when $x \neq 0$, that $R_n = (1-x)^n$. Hence corresponding to any given x' between 0 and 1 we can find n' such that $R_n < \epsilon$ when $n \geqq n'$ for every x in $(x', 1)$. But when $x' \to 0$, then $n' \to \infty$.

It can be proved, as in §§ 182, 183, that Σu_n, when uniformly convergent, can be integrated term by term; and differentiated term by term, if the resulting series is uniformly convergent. The definition and theorems can be immediately extended to series whose terms are functions of more than one variable, or of the complex variable $z = x + iy$.

310. Double power series. Let $a_{mn}x^m y^n$, $m, n = 1, 2, \cdots$, denote an array of elements of the type $a_{mn}x^m y^n$, m indicating the column and n the row in which the element stands.

Suppose that for $x = r(> 0)$ and $y = \rho(> 0)$ every element $a_{mn}x^my^n$ is numerically less than some positive number M. Then for $|x| < r$, $|y| < \rho$ the double series got by taking the elements of the array by rows and adding their sums will be an absolutely convergent series of the type discussed in § 299.

For when $|x| < r$, $|y| < \rho$, then for every m, n,

$$|a_{mn}x^my^n| = |a_{mn}r^m\rho^n| \cdot \left|\frac{x}{r}\right|^m \cdot \left|\frac{y}{\rho}\right|^n < M \cdot \left|\frac{x}{r}\right|^m \left|\frac{y}{\rho}\right|^n$$

Hence the terms of the double series under consideration are numerically less than the corresponding terms of the similar double series got from the array $M\,|x/r|^m \cdot |y/\rho|^n$; and it can be shown, as in § 301 (5), that when $|x| < r$, $|y| < |\rho|$ this latter series converges to the sum

$$M \div \left(1 - \left|\frac{x}{r}\right|\right)\left(1 - \left|\frac{y}{\rho}\right|\right)$$

311. Differential equations. Let $dy/dx = f(x, y)$ be a given differential equation of the first order, and (x_0, y_0) a point in whose neighborhood $f(x, y)$ can be expressed as a power series in $x - x_0$, $y - y_0$. We are to prove that at (x_0, y_0) the equation has a solution of the form $y = \Sigma a_n(x - x_0)^n$, the series having a limit of convergence $l > 0$.

If in $dy/dx = f(x, y)$ we substitute $x - x_0 = x'$, $y - y_0 = y'$ and then drop the accents in the result, we get an equation of the form

$$\frac{dy}{dx} = \Sigma a_{mn}x^my^n \tag{1}$$

If in (1) we make the substitution

$$y = C_1x + C_2x^2 + C_3x^3 + \cdots \tag{2}$$

and then proceed as in § 301 we obtain expressions for the C's which are polynomials in the coefficients a_{mn} of (1). Thus

$$C_1 = a_{00}, \quad C_2 = (a_{10} + a_{00}a_{10})/2, \quad \text{and so on} \tag{3}$$

The series (2) with the coefficients (3) satisfies (1) formally. It is therefore a solution of (1) if it has a limit of convergence $l > 0$.

But assuming that $\Sigma a_{mn}x^my^n$ satisfies the conditions of § 310, and using the notation of that section, consider the equation

$$\frac{dy}{dx} = \Sigma M\left(\frac{x}{r}\right)^m\left(\frac{y}{\rho}\right)^n = \frac{M}{(1 - x/r)(1 - y/\rho)}, \quad |x| < r, |y| < \rho \tag{4}$$

Each coefficient $M/r^m\rho^n$ in the second member of (4) is numerically greater than the corresponding coefficient a_{mn} in (1). It therefore follows from (3) that the series

$$y = C'_1x + C'_2x^2 + C'_3x^3 + \cdots \tag{5}$$

which satisfies (4) formally, as (2) satisfies (1), will have in each of its terms a numerically greater coefficient than (2) has in that term : so that (2) will converge when (5) converges. But the solution at $x = 0$, $y = 0$ of the differential equation formed of the first and third members of (4) is

$$y = \rho - \left[\rho^2 + 2\,M\rho r \log\left(1 - \frac{x}{r}\right)\right]^{1/2}$$

and this can be expanded in a power series in x which will be found to converge when $|x| < r(1 - e^{-k})$, where $k = \rho/2\,Mr$. Therefore, since, by § 186, this power series is identical with (5), the series (2) has a limit of convergence $l > 0$.

It may be added that, with a change in notation, $\Sigma a_{mn}x^my^n$ is the infinite series which the Taylor series of § 222 (6) becomes when $f(x, y)$ and all its partial derivatives are continuous at the point (x_0, y_0) and in its neighborhood and the remainder term $\rightarrow 0$, as $n \rightarrow \infty$. It is not difficult to prove that the sum of this Taylor series is not changed when its terms $a_{mn}x^my^n$ are combined in the manner explained at the top of p. 376.

By an extension of this method it can be proved that a pair of equations $dy/dx = f(x, y, z)$, $dz/dx = \phi(x, y, z)$ has solutions. The equation $d^2y/dx^2 = F(x, y, dy/dx)$ becomes such a pair by setting $dy/dx = z$; hence it has solutions. Similarly for sets of simultaneous equations of the first order in n variables, and equations of the nth order in two variables.

XXVIII. FOURIER SERIES

312. Fourier series. 1. Assume that, for values of x between $-\pi$ and π, a given function $f(x)$ can be expressed by a series of the form

$$f(x) = a_0 + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx) \tag{1}$$

and that it is permissible to integrate this series term by term, as in § 182. We can then show that the coefficients a_0, a_1, b_1, $\cdots$ are given by the formulas

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$
$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx \tag{2}$$

The series (1) with the coefficients given by (2) is called a *Fourier series*.

For it is easy to prove, §§ 112, 115, that

$$\int_{-\pi}^{\pi}\cos mx\,dx = 0 \qquad \int_{-\pi}^{\pi}\sin mx\,dx = 0 \qquad \int_{-\pi}^{\pi}\cos mx \sin nx\,dx = 0$$
$$\int_{-\pi}^{\pi}\cos mx\cos nx\,dx = \int_{-\pi}^{\pi}\sin mx \sin nx\,dx = 0 \qquad (m \neq n)$$
$$\int_{-\pi}^{\pi}dx = 2\pi \qquad \int_{-\pi}^{\pi}\cos^2 mx\,dx = \pi \qquad \int_{-\pi}^{\pi}\sin^2 mx\,dx = \pi$$

Hence, if the equation (1) and the two equations got by multiplying it by $\cos mx$ and by $\sin mx$ be integrated between the limits $x = -\pi$ and $x = \pi$, the results obtained will be

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 2\pi \quad \int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_m\pi \quad \int_{-\pi}^{\pi} f(x)\sin mx\,dx = b_m\pi$$

and these equations give the formulas (2).

EXAMPLE 1. Let $f(x)$ have the value 1 when $-\pi < x < 0$, and the value 2 when $0 < x < \pi$. On the assumption that this function can be expressed by a Fourier series, we have

$$a_0 2\pi = \int_{-\pi}^{0} dx + \int_0^{\pi} 2\,dx = 3\pi$$
$$a_m\pi = \int_{-\pi}^{0}\cos mx\,dx + 2\int_0^{\pi}\cos mx\,dx = 3\int_0^{\pi}\cos mx\,dx = 0$$
$$b_m\pi = \int_{-\pi}^{0}\sin mx\,dx + 2\int_0^{\pi}\sin mx\,dx = \int_0^{\pi}\sin mx\,dx = -\frac{1}{m}\cos mx\Big|_0^{\pi}$$

When m is odd, $\cos mx \Big|_0^\pi = -2$, and when m is even, $\cos mx \Big|_0^\pi = 0$. Hence

$$f(x) = \frac{3}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \cdots$$

Observe that the given $f(x)$ is not defined at $x = -\pi, 0, \pi$, and that it is discontinuous at $x = 0$. The foregoing reckoning tacitly assumes the following definitions of $\int_{-\pi}^{\pi}$, $\int_{-\pi}^{0}$, $\int_0^\pi$ for this $f(x)$ (compare § 134). Using $\alpha, \beta, \gamma, \delta$ to denote positive variables which $\rightarrow 0$, we have

$$\int_{-\pi}^{\pi} = \lim\left[\int_{-\pi+\alpha}^{-\beta} + \int_{\gamma}^{\pi-\delta}\right] = \int_{-\pi}^{0} + \int_0^\pi$$

Example 2. Assuming that (1), (2) apply to $f(x) = x$, show that for any value of x between $-\pi$ and π

$$\frac{x}{2} = \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots$$

2. To prove that the series (1), (2) does actually represent $f(x)$ within the interval $(-\pi, \pi)$, we must show that the sum of its first n terms $\rightarrow f(x)$ when $n \rightarrow \infty$. This proof is given in § 315 under certain restrictions as to the character of $f(x)$. It is based on a theorem which is established in the following two sections.

313. The integral $\int_0^{\frac{\pi}{2}} [\sin(2m+1)x/\sin x]\,dx$. From this integral we shall derive an expression for $\pi/2$ as the sum of an alternating series.

It can be proved by mathematical induction that

$$\frac{1}{2} + \cos\beta + \cos 2\beta + \cdots + \cos m\beta = \sin\frac{2m+1}{2}\beta \Big/ 2\sin\frac{\beta}{2} \quad (1)$$

Hence, setting $\beta = 2x$,

$$\int_0^{\pi/2}\frac{\sin(2m+1)x}{\sin x}dx = 2\int_0^{\pi/2}\left[\frac{1}{2} + \cos 2x + \cdots + \cos 2mx\right]dx = \frac{\pi}{2} \quad (2)$$

1. Setting $2m+1 = n$, and observing that $\sin nx$ changes sign at the points $x = \pi/n, 2(\pi/n), \cdots$, we divide the interval $(0, \pi/2)$ into the subintervals whose end points are

$$0,\ \frac{\pi}{n},\ 2\frac{\pi}{n},\ 3\frac{\pi}{n},\ \cdots,\ r\frac{\pi}{n},\ \frac{\pi}{2} \quad (3)$$

where $r(\pi/n)$ denotes the greatest multiple of π/n which is less than $\pi/2$, and then express the integral (2) as the sum of the integrals correspond-

ing to these $r+1$ subintervals, so that if u_k denote the numerical value of the kth of these integrals we shall have (for $k < r$)

$$(-1)^k u_k = \int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}} \frac{\sin nx}{\sin x}\, dx \tag{4}$$

and
$$\frac{\pi}{2} = u_0 - u_1 + u_2 - \cdots + (-1)^r u_r \tag{5}$$

2. Substituting $x = k\frac{\pi}{n} + y$ in (4) and simplifying, we get

$$u_k = \int_0^{\frac{\pi}{n}} \frac{\sin ny}{\sin [y + k(\pi/n)]}\, dy \tag{6}$$

which shows that *the value of u_k decreases as the subscript k increases.*

3. We shall prove that, when $n \to \infty$, u_k approaches a limit l_k such that

$$\frac{2}{(k+1)\pi} < l_k \leqq \frac{2}{k\pi} \tag{7}$$

For substituting $nx = z$ in (4), we get

$$(-1)^k u_k = \int_{k\pi}^{(k+1)\pi} \frac{\sin z}{n \sin (z/n)}\, dz \tag{8}$$

As n increases, $n \sin \frac{z}{n} = z \frac{\sin(z/n)}{z/n}$ increases and $\to z$.

Hence, as n increases, u_k decreases but remains greater than

$$\left| \int_{k\pi}^{(k+1)\pi} \frac{\sin z}{z}\, dz \right| \quad \therefore > \frac{\left| \int_{k\pi}^{(k+1)\pi} \sin z\, dz \right|}{(k+1)\pi} = \frac{2}{(k+1)\pi}$$

Therefore, § 5, as $n \to \infty$, u_k approaches a limit l_k which is greater than $2/(k+1)\pi$.

Also $\quad u_k < \dfrac{\left| \int_{k\pi}^{(k+1)\pi} \sin z\, dz \right|}{n \sin (k\pi/n)} \quad \therefore l_k \leqq \lim\limits_{n\to\infty} \dfrac{2}{n \sin (k\pi/n)} = \dfrac{2}{k\pi}.$

4. But when $n \to \infty$, then $r \to \infty$. Therefore (5) becomes

$$\frac{\pi}{2} = l_0 - l_1 + l_2 - \cdots + (-1)^k l_k + \cdots \tag{9}$$

the right member being an alternating series in which $l_0 > l_1 > l_2 > \cdots$, by (7); the sum of the terms after $(-1)^k l_k$ is numerically $< l_{k+1}$; and $l_{k+1} \to 0$ when $k \to \infty$.

314. The integral $\int_0^h \phi(x)[\sin nx/\sin x]\, dx$. 1. Let n denote an odd integer, h any positive number $\leqq \pi/2$, and $\phi(x)$ a function which in the interval $(0, h)$ is continuous,

positive and decreasing. And let us represent the integral $\int_0^h \phi(x)\,[\sin nx/\sin x]\,dx$ by $J[\phi(x)]$ or J. We are to prove that

$$\lim_{n\to\infty} J = \phi(0)\,\frac{\pi}{2} \tag{1}$$

As in § 313, 1., divide $(0, h)$ into the subintervals whose end points are $0, \pi/n, \cdots, s(\pi/n), h$, and let v_k be the numerical value of the part of J which belongs to the kth of these subintervals: so that (for $k < s$)

$$(-1)^k v_k = \int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}} \phi(x)\,\frac{\sin nx}{\sin x}\,dx \tag{2}$$

and

$$J = v_0 - v_1 + v_2 - \cdots + (-1)^s v_s \tag{3}$$

By § 282, 2., there is a value x_k of x between $k\frac{\pi}{n}$ and $(k+1)\frac{\pi}{n}$ such that

$$\int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}} \phi(x)\,\frac{\sin nx}{\sin x}\,dx = \phi(x_k)\int_{k\frac{\pi}{n}}^{(k+1)\frac{\pi}{n}} \frac{\sin nx}{\sin x}\,dx \tag{4}$$

Hence, § 313, (4), $\quad v_k = \phi(x_k)u_k \qquad k\frac{\pi}{n} < x_k < (k+1)\frac{\pi}{n} \qquad (5)$

Both $\phi(x_k)$ and u_k decrease as k increases. Hence $v_0 > v_1 > v_2 > \cdots$, and the sum of the terms in (3) after the term $(-1)^k v_k$ is numerically less than the term v_{k+1}.

Hence if we first assign to k any odd value, as great as we please, and then take n great enough to make $k(\pi/n) < h$, we can express (3) in the form

$$J = [\phi(x_0)u_0 - \phi(x_1)u_1 + \cdots - \phi(x_k)u_k] + \theta\phi(x_{k+1})u_{k+1} \quad (0 < \theta < 1) \tag{6}$$

But by § 313, (9), we also have

$$\phi(0)\frac{\pi}{2} = [\phi(0)l_0 - \phi(0)l_1 + \cdots - \phi(0)l_k] + \theta'\phi(0)l_{k+1} \quad (0 < \theta' < 1) \tag{7}$$

Let $n \to \infty$. Then $u_0, u_1, \cdots \to l_0, l_1, \cdots$; also $\phi(x_0), \phi(x_1), \cdots \to \phi(0)$ by (5); therefore the bracketed expression in (6) $\to$ that in (7), and

$$J - \phi(0)\,\frac{\pi}{2} \to \theta''\phi(0)l_{k+1} \qquad (-1 < \theta'' < 1) \tag{8}$$

But we can at the outset take k as great as we please, great enough therefore, by § 313, (7), to make $\phi(0)l_{k+1}$ less than any positive number ϵ that can be assigned. Therefore

$$\lim_{n\to\infty} J = \phi(0)\,\frac{\pi}{2} \tag{9}$$

Observe that throughout this discussion $\phi(0)$ has meant $\lim_{x\to 0}\phi(x)$.

2. The formula (9) also holds good when $\phi(x)$ is a decreasing function which becomes negative in $(0, h)$. For let $-C = \phi(h)$, the least value of $\phi(x)$ in $(0, h)$. Then $\phi(x) + C$ is a positive decreasing function in $(0, h)$. We have

$$J[\phi(x)] = J[\phi(x) + C] - J[C]$$

Hence $\displaystyle \lim_{n \to \infty} J[\phi(x)] = [\phi(0) + C]\frac{\pi}{2} - C\frac{\pi}{2} = \phi(0)\frac{\pi}{2}$

3. Again (9) holds good for a positive or negative function $\phi(x)$ which (when not constant) always increases in $(0, h)$. For $-\phi(x)$ is then a decreasing function; hence

$$\lim_{n \to \infty} J[\phi(x)] = -\lim_{n \to \infty} J[-\phi(x)] = -\left[-\phi(0)\frac{\pi}{2}\right] = \phi(0)\frac{\pi}{2}$$

4. Let g and h be any numbers such that $0 < g < h \leqq \pi/2$, and let $\phi(x)$ denote any continuous function which (when not constant) always decreases or always increases in (g, h); then

$$\lim_{n \to \infty} \int_g^h \phi(x) \frac{\sin nx}{\sin x} dx = 0 \tag{10}$$

For let $\phi_1(x)$ denote a function which has the constant value $\phi(g)$ in $(0, g)$ and coincides with $\phi(x)$ in (g, h); then

$$\int_g^h \phi(x) \frac{\sin nx}{\sin x} dx = \int_0^h \phi_1(x) \frac{\sin nx}{\sin x} dx - \int_0^g \phi_1(x) \frac{\sin nx}{\sin x} dx$$

When $n \to \infty$ both the integrals on the right $\to \phi_1(0)\pi/2$, by (9); hence that on the left $\to 0$.

5. Thus far we have supposed $h \leqq \pi/2$. Let us now suppose $h > \pi/2$ but $< \pi$. Break up the integral J into the part belonging to $(0, \pi/2)$ and the part belonging to $(\pi/2, h)$, and in the second part set $x = \pi - y$. We obtain

$$J = \int_0^{\pi/2} \phi(x) \frac{\sin nx}{\sin x} dx + \int_{\pi - h}^{\pi/2} \phi(\pi - y) \frac{\sin ny}{\sin y} dy$$

When $n \to \infty$, the first integral $\to \phi(0)\pi/2$, the second $\to 0$. Hence (9) holds for any $\phi(x)$ which varies in but one sense in $(0, h)$, $h < \pi$.

6. *Suppose* $0 < h < \pi$, *and let* $\phi(x)$ *be a function for which the interval* $(0, h)$ *can be divided into a finite number of parts* $(0, a)$, (a, b), $\cdots$ (g, h) *within each of which* $\phi(x)$ *is continuous and varies in but one sense; then*

$$\lim_{n \to \infty} J = \lim_{n \to \infty} \int_0^h \phi(x) \frac{\sin nx}{\sin x} dx = \phi(0) \frac{\pi}{2} \tag{11}$$

For break J up into the parts belonging to $(0, a)$, (a, b), $\cdots$. By (9), (10), when $n \to \infty$, the first of these parts $\to \phi(0)\pi/2$ and the others $\to 0$.

The proof holds good even when at any point of division, as a, we have $\lim\limits_{x \to a} \phi(x) \neq \lim\limits_{x \to a} \phi(x)$, so that $\phi(x)$ is discontinuous at $x = a$.

315. Proof of Fourier's theorem. Let $f(x)$ denote a function which is finite in the interval $(-\pi, \pi)$, and is such that this interval can be divided into a finite number of parts within each of which $f(x)$ is continuous and varies in but one sense. If at any point of division, $x = c$, we have $\lim\limits_{x \to c} f(x) \neq \lim\limits_{x \to c} f(x)$, we represent these limits by $f(c - 0)$ and $f(c + 0)$ and then define $f(c)$ by the formula

$$f(c) = \frac{f(c - 0) + f(c + 0)}{2}$$

Obviously this formula is true identically at any point where $f(x)$ is continuous.

We are to prove that for all values of x between $-\pi$ and π this function $f(x)$ may be represented by the Fourier series of § 312, and that when x is $-\pi$ or π the sum of the series is

$$\frac{f(-\pi + 0) + f(\pi - 0)}{2}$$

For taking α as the variable of integration, we may write the kth term under the Σ sign in § 312 (1), with the coefficients given by § 312 (2), in the form

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)[\cos k\alpha \cos kx + \sin k\alpha \sin kx]\, d\alpha = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha) \cos k(\alpha - x)\, d\alpha$$

Hence, if S_m denote the sum of the first $m+1$ terms of the series, we have

$$S_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)[\tfrac{1}{2} + \cos(\alpha - x) + \cdots + \cos m(\alpha - x)]\, d\alpha \qquad (1)$$

We shall prove that when x is between $-\pi$ and π, then $\lim S_m = f(x)$.

Apply the formula § 313 (1); then set $\alpha - x = 2y$, $2m + 1 = n$.

We thus obtain
$$S_m = \frac{1}{\pi}\int_{-\frac{\pi+x}{2}}^{\frac{\pi-x}{2}} f(x + 2y)\frac{\sin ny}{\sin y}\, dy \qquad (2)$$

Separate this integral into one between $-(\pi + x)/2$ and 0, and one between 0 and $(\pi - x)/2$, and in the first of these integrals set $y = -z$. We thus get

$$S_m = \frac{1}{\pi}\int_0^{\frac{\pi+x}{2}} f(x - 2z)\frac{\sin nz}{\sin z}\, dz + \frac{1}{\pi}\int_0^{\frac{\pi-x}{2}} f(x + 2y)\frac{\sin ny}{\sin y}\, dy \quad (3)$$

When $-\pi < x < \pi$, both $\dfrac{\pi + x}{2}$ and $\dfrac{\pi - x}{2}$ are between 0 and π. Hence, by § 314 (11),

$$\lim_{m\to\infty} S_m = \frac{1}{\pi}f(x - 0)\frac{\pi}{2} + \frac{1}{\pi}f(x + 0)\frac{\pi}{2} = \frac{f(x - 0) + f(x + 0)}{2} = f(x) \quad (4)$$

This conclusion does not follow when $x = \pi$ or $-\pi$. But when $x = -\pi$, (2) becomes

$$S_m = \frac{1}{\pi}\int_0^{\pi} f(-\pi + 2y)\frac{\sin ny}{\sin y}\, dy \qquad (5)$$

Separate this integral into one between 0 and $\pi/2$, and one between $\pi/2$ and π, and in the second of these integrals set $y = \pi - z$. We thus obtain

$$S_m = \frac{1}{\pi}\int_0^{\pi/2} f(-\pi + 2y)\frac{\sin ny}{\sin y}\, dy + \frac{1}{\pi}\int_0^{\pi/2} f(\pi - 2z)\frac{\sin nz}{\sin z}\, dz \quad (6)$$

Hence, § 314 (11), $\displaystyle\lim_{m\to\infty} S_m = \frac{f(-\pi + 0) + f(\pi - 0)}{2}$ (7)

We obtain the same result for $x = \pi$.

316. Convergence of Fourier series. The Fourier series for $f(x)$ has been shown to be convergent; but, generally speaking, it is only conditionally not absolutely convergent. In case $f(x)$ has points of discontinuity in $(-\pi, \pi)$, the series cannot be uniformly convergent, § 309, but one can prove that it is uniformly convergent if $f(x)$ is continuous in $(-\pi, \pi)$.

317. Fourier series in cosines or sines only. It is readily seen that if $f(x)$ is an even function, that is, if $f(-x) = f(x)$, then

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)dx \qquad a_m = \frac{2}{\pi}\int_0^\pi f(x)\cos mx\,dx \qquad b_m = 0 \quad (1)$$

Similarly, if $f(x)$ is an odd function, so that $f(-x) = -f(x)$,

$$a_m = 0 \qquad\qquad b_m = \frac{2}{\pi}\int_0^\pi f(x)\sin mx\,dx. \qquad (2)$$

It follows that any function $f(x)$ which satisfies the conditions of § 314, 6. in the interval $(0, \pi)$ may be expressed in that interval by a Fourier series in cosines only, or sines only, with the coefficients given by (1) or (2). For let $F(x)$ denote the function which equals $f(x)$ for any value of x in $(0, \pi)$, and also for the negative of that value. This $F(x)$ is an even function; the formulas (1) give its series in $(-\pi, \pi)$; and in $(0, \pi)$, $F(x)$ is $f(x)$. Similarly for the case of the sine series.

EXERCISE LXII

1. Find the Fourier series for the following in the interval $(-\pi, \pi)$

1. $x^2 + x$ 2. $1 + \sin x$ 3. $x \sin x$

2. Express $\pi/4$ by a sine series and also by a cosine series in the interval $(0, \pi)$.

3. Express $\sin x$ by a cosine series in the interval $(0, \pi)$.

4. If $f(x)$ has the value -1 when $0 < x < \pi/2$, and the value 1 when $\pi/2 < x < \pi$, find the sine series for $f(x)$ in the interval $(0, \pi)$.

5. Find the cosine series in $(0, \pi)$ for $\cos ax$ when a is not an integer.

XXIX. PROPERTIES OF CONTINUOUS FUNCTIONS

THE SET OF REAL NUMBERS

318. The rational numbers. 1. *The infinite set of symbols called rational numbers is an ordered or "ordinal" set.*

We have first of all the *natural scale* 1, 2, 3, 4, 5, ⋯, consisting of the natural numbers arranged as they are used in counting. Next, to make subtraction of natural numbers always possible, this scale is extended backward. We invent successively the symbol 0 which we place before 1, the symbol -1 which we place before 0, and so on without end. We thus obtain the *complete scale*

$$\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots$$

Also, to make division of natural numbers always possible, we invent *fractions*. Taking any pair of natural numbers, a and b, of which a, but not b, may also be 0, we form the symbol a/b, read "a over b," with the understanding that $a/1 = a$. We then think of all such symbols as inserted among the numbers of the scale $0/1, 1/1, 2/1, 3/1, \cdots$ in such an order that if a/b and c/d denote any two of them, a/b shall precede, coincide with, or follow c/d, according as in the scale the number ad precedes, coincides with, or follows the number bc: or, employing $<, =, >$ to mean "precedes," "coincides with," "follows," so that

$$a/b <, =, \text{ or } > c/d \text{ according as } ad <, =, \text{ or } > bc$$

Finally we assign to $-a/b$ a position relative to the numbers of the complete scale which is symmetric with respect to 0 to that of a/b.

2. *Between any two rationals there are infinitely many other rationals. The set of rational numbers is therefore said to be "dense."*

319. Irrational numbers. The rational numbers meet all the requirements of the four fundamental operations of arithmetic. But other requirements suggest the invention of additional numbers. There is, for example, no rational whose square is 2, none therefore that will satisfy the equation $x^2 - 2 = 0$ or express the length of the diagonal of a square in terms of the side.

In relation to any particular rational, say 1/2, all other rationals naturally fall into two classes, the one consisting of all that precede 1/2, the other of all that follow 1/2. Call these two classes of numbers C_1 and C_2. The relations existing between them are these :

1. Each number in C_1 precedes every number in C_2.
2. There is no last number in C_1 and no first number in C_2.

As for 1/2 itself, it is the one number between all numbers in C_1 and all numbers in C_2.

Let us now consider the problem of defining a number whose square shall be 2. Since there is no rational whose square is 2, every rational is either a number whose square is less than 2 or one whose square is greater than 2. Assign all positive rationals whose squares are greater than 2 to a class A_2, and all other rationals to a class A_1.

1. Each number in A_1 precedes every number in A_2.
2. There is no last number in A_1 and no first in A_2.

Hence the classes A_1 and A_2 stand in precisely the same relation to each other as did the classes C_1 and C_2 in the case just considered. But since all the rationals are in A_1 and A_2 themselves, no number as yet exists lying between A_1 and A_2, as 1/2 lies between C_1 and C_2. We invent such a number, defining it ordinally as the one number which lies between all the numbers in A_1 on the one hand, and all in A_2 on the other.

When multiplication has been defined for this new number and others of the same kind, we shall find that its square is 2, and that it may therefore be represented by the symbol $\sqrt{2}$. Hence, by definition, "$\sqrt{2}$ is that number which lies between all positive rationals whose squares are less than 2 and all whose squares are greater than 2."

We call the number $\sqrt{2}$ thus defined an *irrational number.*

An irrational number a is defined whenever a law is stated which assigns every given rational to one and but one of two classes A_1, A_2 so related that (1) *each number in A_1 precedes every number in A_2, and* (2) *there is no last number in A_1 and no first in A_2; the definition of a then being: a is the one number between all numbers in A_1 and all in A_2. The definition supposes that there are numbers in both A_1 and A_2.*

Think of every such irrational a as inserted among the rationals in the place given it by its definition. The entire set of numbers, rational and irrational, thus obtained is called the set of *real numbers.*

It is easily proved that the set of real numbers is ordinal and dense. It also possesses the following property, — which the rational set does not possess.

320. Theorem. *The set of real numbers is "continuous": that is, if the entire set be separated into any two parts R_1 and R_2 so related that each number in R_1 precedes every number in R_2, there is always either a last number in R_1 or a first in R_2, but never both.*

For let A_1 and A_2 be the sets of rationals in R_1 and R_2.

If there be a last number in A_1, it must also be the last in R_1; for between it and any number of R_1 supposed to follow it there would be rationals, in R_1 but not in A_1, which is impossible.

Similarly, if there be a first number in A_2, it must also be the first in R_2.

If there be neither a last in A_1 nor a first in A_2, there is an irrational a between the numbers A_1 and the numbers A_2, § 319. By hypothesis,

a belongs either to R_1 or to R_2. If a belongs to R_1, it is the last number in R_1; for between it and any number of R_1 supposed to follow it there would be rationals, in R_1, but not in A_1, which is impossible. Similarly if a belongs to R_2, it is the first number in R_2.

Hence in every case there is either a last number in R_1 or a first in R_2; and there cannot be both since between them there would be rationals, in neither A_1 nor A_2, which is impossible.

321. Approximate values of irrationals. *Let a denote a given irrational. If any positive rational ϵ be assigned, however small, it is always possible to find rationals a_1, a_2, such that*

$$a_1 < a < a_2 \quad \textit{and} \quad a_2 - a_1 \gtreqless \epsilon$$

For if we take any rational $< a$ and repeatedly add ϵ, we shall ultimately reach such a pair of rationals a_1, a_2.

We call a_1 and a_2 *approximate values* of a to less than ϵ.

322. The number $-a$. Let a denote any given irrational, and a_1, a_2 any rationals whatsoever such that $a_1 < a < a_2$.

Consider the sets of numbers $-a_2$ and $-a_1$. Every rational belongs to one or the other of these sets; we have always $-a_2 < -a_1$, and there is no last $-a_2$ nor first $-a_1$. Hence there is a single number, irrational, between all numbers $-a_2$ and all numbers $-a_1$. We call it $-a$.

323. The number $1/a$. Let a denote any given irrational, and a_1, a_2 any rationals to the same side of 0 as a and such that $a_1 < a < a_2$. By the reasoning in § 322, there is a single number, irrational, between all numbers $1/a_2$ and all numbers $1/a_1$. We call this number $1/a$.

324. Addition and subtraction. Let a and b denote any two real numbers, and a_1, a_2 and b_1, b_2 any rationals such that

$$a_1 < a < a_2 \qquad\qquad b_1 < b < b_2$$

We have always $a_1 + b_1 < a_2 + b_2$; and there is no last $a_1 + b_1$ nor first $a_2 + b_2$. Hence there is at least one number

between all numbers $a_1 + b_1$ and all numbers $a_2 + b_2$, and it can be shown that there is but one.[1] We call it $a + b$.

We define $a - b$ by the formula $a - b = a + (-b)$.

325. Multiplication and division. Let a and b be any positive real numbers, and a_1, a_2, b_1, b_2 any positive rationals such that $a_1 < a < a_2$, and $b_1 < b < b_2$. By the reasoning in § 324, there is a single number[2] between all numbers a_1b_1 and all numbers a_2b_2. We call this number ab.

We define a/b by the formula $a/b = a(1/b)$.

Also, by definition, we make $a \cdot 0 = 0$, $a(-b) = -ab$, and so on.

326. Measurement. Let s denote a unit line segment, S a segment incommensurable with s, and a_1 and a_2 the rationals which are the lengths, in terms of s, of any two segments, S_1 and S_2, commensurable with s, and less, and greater than S. There is a single number, an irrational a, between all the numbers a_1 and all the numbers a_2; we call a the *length of S in terms of s*. Thus the ratio of the diagonal to the side of a square is $\sqrt{2}$.

It is assumed conversely (*axiom of continuity*) that to any given positive irrational a corresponds an S whose length in terms of s is a.

It may be proved that if the lengths of S and T in terms of s are a and b, then the length of S in terms of T is a/b.

327. Bounded sets of numbers. *Let A denote a given "bounded set of numbers," that is, a set all numbers of which are between two definite finite numbers c and c'.*

[1] Were there more than one, there would be rationals between them, and if d were the difference of two of these rationals, we should have $(a_2 + b_2) - (a_1 + b_1) > d$ for every a_1, a_2 and b_1, b_2. But by § 321, we can take a_1, a_2 and b_1, b_2 such that $a_2 - a_1 < d/2$, $b_2 - b_1 < d/2$ $\therefore (a_2 + b_2) - (a_1 + b_1) < d$.

[2] Were there more than one, we could find $d > 0$ such that always $a_2b_2 - a_1b_1 > d$. But take any a_2 and call it a_2', then take b_1, b_2 such that $b_2 - b_1 < d/2\,a_2'$, and using this b_2, take a_1, a_2 such that $a_2 - a_1 < d/2\,b_2$. We then have $a_2b_2 - a_1b_1 = (a_2 - a_1)b_2 + (b_2 - b_1)a_1 < d/2 + a_1d/2\,a_2' < d$.

1. *There is a number M which is either the greatest of the different numbers of A, or the least of all numbers which are greater than those of A. We call M the "least upper bound" of A.*

2. *There is a number m which is either the least of the different numbers of A or the greatest of all numbers which are less than those of A. We call m the "greatest lower bound" of A.*

For let R_2 be the set of numbers greater than those of A, and R_1 the rest of the real system. By § 320, there is either a greatest number in R_1 or a least in R_2. This number is the least upper bound M.

The existence of m may be proved in a similar manner.

3. *Except when M is an isolated number of A, there are numbers of A between M and every number $M - \epsilon$ less than M. Similarly for m.*

Example. The set $2, 2\frac{1}{2}, 2\frac{3}{4}, 2\frac{7}{8}, \cdots$ has the least upper bound 3 and the greatest lower bound 2. There are numbers of the set between 3 and $3 - \epsilon$. But 2 is isolated.

328. Limits. 1. *If a variable x continually increases but remains less than some definite number c, it approaches a limit; and this limit is either c or some lesser number.*

2. *If x continually decreases but remains greater than some definite finite number c, it approaches a limit; and this limit is either c or some greater number.*

For in Case 1. the values taken by x have a least upper bound M; and since x remains less than M but will ultimately become greater than every number $M - \epsilon$ which is less than M, it approaches M as limit, § 4.

Similarly in Case 2., $x \to m$, the greatest lower bound of the values of x.

3. *If x increases and y decreases, x remains less than y, and $y - x \to 0$, then x and y approach the same number as limit.*

For, by 1., 2., x and y approach limits, a and b, such that $x < a \leqq b < y$. But we cannot have $a < b$, since in that case $y - x$ would not become less than $b - a$, whereas by hypothesis $y - x \to 0$. Hence $a = b$.

EXAMPLE. The never ending sequence $x_1, x_2, \cdots, x_p, \cdots x_{p+n}, \cdots$ is supposed to have the property that to any given positive number δ corresponds a term x_p such that $|x_{p+n} - x_p| < \delta$ for $n = 1, 2, 3, \cdots$. Let M_p and m_p denote the least upper and greatest lower bounds of the numbers x_{p+n} and show that as $\delta \to 0$, M_p, m_p, and therefore x_p, approach a common limit.

PROPERTIES OF CONTINUOUS FUNCTIONS

329. Oscillation of a function $f(x)$. Continuous functions. 1. Let the set of the values of the function $f(x)$ in the interval (a, b) be a bounded set, § 327, and let M and m be the least upper and greatest lower bounds of this set. We call $M - m$ the *oscillation* of $f(x)$ in (a, b).

Using this notion, we may replace the definition of a continuous function in § 16 by that which follows.

2. The function $f(x)$ is said to be continuous at the point $x = c$ when $f(c)$ has a definite value and when to every given positive number ϵ corresponds a second positive number δ such that in every interval of length δ which contains c the oscillation of $f(x)$ is less than ϵ.

A point c where these conditions are not both satisfied is called a point of discontinuity of $f(x)$.

We say that $f(x)$ is continuous in the interval (a, b) if it is continuous at every point of (a, b), it being sufficient at a and b that there be intervals (a, x') and (x'', b) in which the oscillation of $f(x)$ is $< \epsilon$.

3. Since the sets of numbers x such that

$$a < x \leqq b, \quad \text{or} \quad a \leqq x < b, \quad \text{or} \quad a < x < b$$

are sometimes called *open intervals* (a, b), we shall now call the interval (a, b) defined in § 2 a *closed interval.*

330. Theorem 1. *Let $f(x)$ be continuous in the closed interval (a, b), and let ϵ denote any given positive number,*

however small. It is always possible to divide (a, b) *into parts in each of which the oscillation of* $f(x)$ *is less than* ϵ.

For divide (a, b) into two equal parts, then each of these parts into two equal parts, and so on. The process must ultimately yield parts in each of which the oscillation of $f(x)$ is less than ϵ.

For if not, it must be the case that the oscillation of $f(x)$ is not less than ϵ in at least one of the halves of (a, b), call it (a_1, b_1), and again in at least one of the halves of (a_1, b_1), call it (a_2, b_2), and so on without end. Let (a_n, b_n) be the nth subinterval thus obtained. As n increases, a_n increases (or is stationary), b_n decreases (or is stationary), and $b_n - a_n = (b - a)/2^n \to 0$; hence, § 328, 3., a_n and b_n approach a common limit c. And since c is in (a_n, b_n), and by the supposition made the oscillation of $f(x)$ in (a_n, b_n) never becomes less than ϵ however great n is taken, $f(x)$ is not continuous at c, § 329, 2.

But this conclusion contradicts the hypothesis that $f(x)$ is continuous in (a, b), and therefore at c. Hence the process of dividing (a, b) into halves, and so on, must ultimately yield parts in each of which the oscillation of $f(x)$ is less than ϵ.

331. Theorem 2. *If* $f(x)$ *is continuous in the closed interval* (a, b), *it is "uniformly continuous" in* (a, b): *that is, to any given positive number* ϵ *corresponds another positive number* δ *such that in every part of* (a, b) *of length* δ *the oscillation of* $f(x)$ *is less than* ϵ.

For, by Theorem 1., we can divide (a, b) into equal parts in each of which the oscillation of $f(x)$ is less than $\epsilon/2$. Let δ denote the length of one of these parts, and let (x', x'') denote *any* part of (a, b) of length δ. Since (x', x'') can overlap at most two of the consecutive equal parts, the oscillation of $f(x)$ in (x', x'') is less than $2(\epsilon/2)$ or ϵ.

EXAMPLE. Prove that in the open interval $0 < x \leqq 1$, the function $f(x) = 1/x$ is continuous but not uniformly continuous.

332. Theorem 3. *If $f(x)$ is continuous in the closed interval (a, b), its values in (a, b) have a least upper bound M and a greatest lower bound m. We call M and m the least upper and greatest lower bounds of $f(x)$ in (a, b).*

For since we can divide (a, b) into 2^n equal parts in each of which the oscillation of $f(x)$ is less than ϵ, the oscillation of $f(x)$ in (a, b) itself is less than $2^n\epsilon$. Hence all the values of $f(x)$ in (a, b) are between the two finite members $f(a) - 2^n\epsilon$ and $f(a) + 2^n\epsilon$; they therefore have least upper and greatest lower bounds M and m, § 327.

333. Theorem 4. *If $f(x)$ is continuous in the closed interval (a, b), it has a greatest and a least value in (a, b): that is, there exist values, x_1 and x_2, of x in (a, b) such that $f(x_1) = M$ and $f(x_2) = m$.*

For divide (a, b) into parts as in the proof of Theorem 1. Then M is the least upper bound of $f(x)$ in at least one of the halves of (a, b), call it (a_1, b_1), and again in at least one of the halves of (a_1, b_1), call it (a, b_2), and so on without end. Let (a_n, b_n) denote the nth subinterval thus obtained, and let x_1 denote the limit which both a_n and b_n approach, as $n \rightarrow \infty$. Then $f(x_1) = M$.

For if this be not true, then since M and $f(x_1)$ have definite values, $M - f(x_1)$ must be some definite positive number d. But this is impossible since, x_1 being a number in (a_n, b_n), $M - f(x_1)$ cannot exceed the oscillation of $f(x)$ in (a_n, b_n); and $f(x)$ being continuous at $x = x_1$, we can take n great enough to make the oscillation of $f(x)$ in (a_n, b_n) less than d, § 329, 2. Hence $d = 0$, that is, $f(x_1) = M$.

Similarly there is a number x_2 in (a, b) such that $f(x_2) = m$.

334. Theorem 5. *If $f(x)$ is continuous in the closed interval (a, b), and $f(a)$ and $f(b)$ have opposite signs, there is a value x_0 of x in (a, b) such that $f(x_0) = 0$.*

Suppose that $f(a)$ is $+$ and $f(b)$ is $-$. Divide (a, b) into parts as in the proof of Theorem 1. Then, unless $f(x)$ is 0 at one of the points of division, — in which case the theorem is true, — one of the halves of (a, b), call it (a_1, b_1), is such that $f(a_1)$ is $+$ and $f(b_1)$ is $-$; again one of the halves of (a_1, b_1), call it (a_2, b_2), is such that $f(a_2)$ is $+$ and $f(b_2)$ is $-$; and so on without end. Let (a_n, b_n) denote the nth subinterval obtained by this process, and let x_0 denote the limit which both a_n and b_n approach, as $n \to \infty$. Then $f(x_0) = 0$.

For since $f(x)$ is continuous at $x = x_0$, we have

$$f(x_0) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n)$$

But since $f(a_n)$ is always $+$, $\lim\limits_{n\to\infty} f(a_n)$ cannot be $-$; and since $f(b_n)$ is always $-$, $\lim\limits_{n\to\infty} f(b_n)$ cannot be $+$. Hence both of these limits, and therefore $f(x_0)$, must be 0.

335. Theorem 6. *If $f(x)$ is continuous in the closed interval (a, b), then between any two values, x_1 and x_2, of x in (a, b), $f(x)$ takes every value between $f(x_1)$ and $f(x_2)$.*

For let C denote any given number between $f(x_1)$ and $f(x_2)$. The function $f(x) - C$ is continuous in the interval (x_1, x_2) and has opposite signs at the ends x_1 and x_2 of the interval. Hence, by Theorem 5, there is a value x_0 of x in (x_1, x_2) such that

$$f(x_0) - C = 0 \qquad \text{or} \qquad f(x_0) = C$$

EXAMPLE 1. Show that if $f(x)$ is continuous in (a, b) it takes every value between its least and greatest values in (a, b).

EXAMPLE 2. By applying Theorem 6 to $f(x) = x^n - a$, prove that the set of real numbers contains the nth root of any positive number a.

FUNCTIONS OF TWO OR MORE VARIABLES

336. Oscillation of $f(x, y)$. Let S denote a given enclosed region in the xy-plane (Fig. 147); let the values of $f(x, y)$ at the points of S, the contour points included, be a bounded

set; and let M and m denote the least upper and greatest lower bounds of this set: we call $M - m$ the *oscillation of* $f(x, y)$ in S.

337. Continuous functions. Using the notion of oscillation, we may replace the definition of a continuous function $f(x, y)$ given in § 148 by that which follows.

1. The function $f(x, y)$ is said to be continuous at the point $P(a, b)$ of the region S, when $f(x, y)$ has a definite value at P and when to every given positive number ϵ corresponds a second positive number ρ such that in the part of S within every circle of radius ρ that encloses P the oscillation of $f(x, y)$ is less than ϵ.

A point P where these conditions are not both satisfied is called a point of discontinuity of $f(x, y)$.

2. If $f(x, y)$ is continuous at all points of S, contour points included, then $f(x, y)$ is said to be *continuous in* S.

338. Theorem 7. *Let $f(x, y)$ be continuous within S and on its contour, and let ϵ denote any given positive number. It is possible to divide S into parts in each of which the oscillation $f(x, y)$ is less than ϵ.*

For, as in Fig. 147, enclose S in a rectangle R whose sides are parallel to Ox and Oy. Then by parallels to Ox, Oy divide R into four equal rectangles. Deal similarly with such of these quarter rectangles as contain points of S, and so on. This process must ultimately yield parts of S in each of which the oscillation of $f(x, y)$ is less than ϵ.

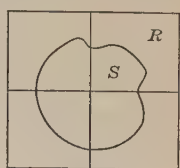

FIG. 147.

For, if not, there is a quarter of R, call it R_1, which contains a part of S in which the oscillation of $f(x, y)$ is not less than ϵ; the like is true of one of the quarters of R_1, call it R_2, and so on without end. But this is impossible. For if R_n denote the nth rectangle thus obtained, it is readily shown, by § 328, 3., that as $n \to \infty$, all the points of R_n approach a definite point (a, b) in R_n as limit. And were it true that, as $n \to \infty$, the oscillation of $f(x, y)$ in the part of S in R_n re-

mained $\gtreqless \epsilon$, then, by § 337, 1., $f(x, y)$ would not be continuous at (a, b); whereas, by hypothesis, $f(x, y)$ is continuous at all points of S and therefore at (a, b) which is in S.

Hence this process must ultimately yield parts of S in each of which the oscillation of $f(x, y)$ is less than ϵ.

From this theorem, by the reasoning in §§ 331–333, we deduce the following:

339. Theorem 8. *If $f(x, y)$ is continuous within S and on its contour, it is "uniformly continuous" in S: that is, to every given positive number ϵ corresponds another positive number ρ such that in every part of S which can be enclosed in a circle of radius ρ the oscillation of $f(x, y)$ is less than ϵ.*

340. Theorem 9. *If $f(x, y)$ is continuous within S and on its contour, it has a greatest and a least value in S: that is, the set of values of $f(x, y)$ in S is a bounded set, and if M and m denote the least upper and greatest lower bounds of the set, there are points (x_1, y_1) and (x_2, y_2) within S or on its contour such that $f(x_1, y_1) = M$ and $f(x_2, y_2) = m$.*

This discussion may be extended to functions of more than two variables, as $f(x, y, z)$, $f(x, y, z, w)$, and yields theorems for such functions which are the counterparts of those just proved.

XXX. FUNCTIONS OF A COMPLEX VARIABLE

341. Complex numbers. There being no real number whose square is -1, we invent a new number which we call the *unit of imaginaries* and represent by i, subsequently formulating a definition of multiplication according to which $i^2 = -1$ and therefore $i = \sqrt{-1}$.

Then, taking any pair of real numbers, a and b, we form symbols of the type $a + bi$, which we call *complex numbers*, with the understanding that by definition

1. $a + 0\,i = a$ 2. $0 + bi = bi$ 3. $1\,i = i$

so that the set of complex numbers contains the set of real numbers a and the set of *pure imaginaries* bi, including i; and that

4. $a + bi = c + di$ is to mean $a = c$ and $b = d$

342. The fundamental operations. If we assume that the elementary rules for reckoning with complex numbers are the same as for real numbers, except that $i^2 = -1$, and then seek to express the sum, difference, product, and quotient of two complex numbers, each in the form of a single complex number, we obtain :

5. $(a + bi) + (c + di) = (a + c) + (b + d)i$

6. $(a + bi) - (c + di) = (a - c) + (b - d)i$

7. $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

8. $$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\,i$$

the last being got by multiplying numerator and denominator of $(a + bi)/(c + di)$ by $c - di$ and simplifying the result.

We make the formulas 5.–8. our *definitions* of the sum, difference, product, and quotient of $a + bi$ and $c + di$. To justify them, it is only necessary to show that they are consistent with 1., 2., 3. and the corresponding definitions for real numbers — which is easily done. That 7. gives $i^2 = -1$, we find by setting $a = c = 0$, $b = d = 1$, and applying 1.–3.

343. Graphical representation. 1. Any complex number $\alpha = a + bi$ being given, take rectangular axes Ox, Oy, and plot the point A whose coordinates are (a, b). We may represent α by this point A.

2. We call $r = (a^2 + b^2)^{1/2}$ the *absolute value* of α and write

$$|\alpha| = |a + bi| = (a^2 + b^2)^{1/2}.$$

We call $\theta = xOA$ the *angle of* α. Since $a = r \cos\theta$, $b = r \sin\theta$,

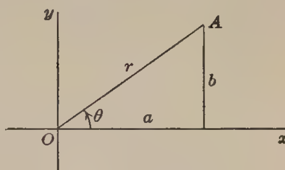

FIG. 148.

$$a + bi = r(\cos\theta + i \sin\theta) \qquad (1)$$

3. We may also represent $a + bi$ by the vector OA. The factor r of (1) is the length of OA. We may call $\cos\theta + i\sin\theta$ the *direction factor* of OA, or of $a + bi$.

344. Sums and differences. Let A and B represent $a + bi$ and $c + di$, and construct the parallelogram $OACB$. The point C represents

$$a + bi + (c + di). \quad \text{For}$$

$$OF = OE + EF = OE + OD = a + c$$

$$FC = FG + GC = EA + DB = b + d$$

Or, representing $\alpha = a + bi$ and $\beta = c + di$ by the vectors OA, OB, then, § 83,

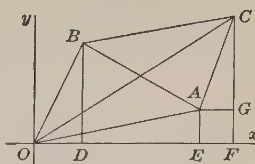

FIG. 149.

$$\alpha + \beta = OA + OB = OA + AC = OC \qquad (2)$$

We also have by subtraction of vectors, § 83 (3),

$$\alpha - \beta = OA - OB = BA \qquad (2)$$

We may get the *point* $\alpha - \beta$ by shifting the initial point of B of BA to O.

If $\overline{OC}$ denote the length of the vector OC, and so on, we have $\overline{OC} < \overline{OA} + \overline{AC} = \overline{OA} + \overline{OB}$. Hence

$$|\alpha + \beta| \leqq |\alpha| + |\beta| \tag{3}$$

the sign = being taken when OA and OB have the same direction.

345. Products, powers, quotients. 1. By aid of the formulas for $\cos(\theta + \theta')$ and $\sin(\theta + \theta')$ in terms of $\sin\theta$, $\cos\theta$, $\sin\theta'$, $\cos\theta'$, we obtain

$$r(\cos\theta + i\sin\theta)\cdot r'(\cos\theta' + i\sin\theta') = rr'[\cos(\theta + \theta') + i\sin(\theta + \theta')] \tag{4}$$

The absolute value of the product of two complex numbers is the product of their absolute values, and its angle is the sum of their angles.

The same is true of a product of n factors, as may be shown by repeated applications of (4). When these n factors are equal, we have

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) \tag{5}$$

which (when $r = 1$) is called *De Moivre's theorem.*

2. The *quotient* of $r(\cos\theta + i\sin\theta)$ by $r'(\cos\theta' + i\sin\theta')$ is

$$(r/r')[\cos(\theta - \theta') + i\sin(\theta - \theta')] \tag{6}$$

For (6), when multiplied by $r'(\cos\theta' + i\sin\theta')$, gives $r(\cos\theta + i\sin\theta)$.

3. The n *nth roots* of $r(\cos\theta + i\sin\theta)$ are the n numbers

$$r^{1/n}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right), k = 0, 1, 2, \cdots, n - 1 \tag{7}$$

For, by (5), the nth power of each of these n numbers is $r(\cos\theta + i\sin\theta)$.

EXERCISE LXIII

1. Construct the points which represent the square roots of $4 + 3i$.

2. Find the points which represent the roots of $x^3 - i = 0$; of $x^5 + 32 = 0$.

3. Show that $(a + ib)(c + id)$ is 0 when, and only when, $a + ib$ or $c + id$ is 0.

4. By De Moivre's theorem show that $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.

5. Show, by (7), that the three cube roots of 1 are $1, (-1 \pm i\sqrt{3})/2$.

6. Since $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$, and $\sin 3\theta$ vanishes when $\theta = 0$ or $\pm \pi/3$, show that $\sin 3\theta = -4 \sin \theta\, [\sin^2\theta - \sin^2(\pi/3)]$.

7. Show that when n is odd, $\sin n\theta$ equals a polynomial of the nth degree in $\sin \theta$ which vanishes when $\theta = 0, \pm \frac{\pi}{n}, \cdots, \pm \frac{n-1}{2} \frac{\pi}{n}$.

346. The complex variable. When x and y are real variables, we call $z = x + iy$ a *complex variable.*

Let $\alpha = a + ib$ denote a given constant, and suppose that z varies in such a manner that $|z - \alpha|$, the distance of the point z from the point α, approaches the limit 0; we then say that *z approaches α as limit:* in other words,

$$z \to \alpha \qquad \text{means} \qquad |z - \alpha| \to 0 \tag{1}$$

The statement $z \to \alpha$ implies $x \to a$, $y \to b$; and conversely.

The theorems respecting limits of sums, products, and quotients of real numbers, § 7, *hold good for complex numbers.*

For they depend only on the definition (1) and the relations $|\alpha + \beta| \leqq |\alpha| + |\beta|$, $|\alpha\beta| = |\alpha| \cdot |\beta|$, which hold good for complex as for real numbers.

347. Rational functions. By aid of the fundamental operations, § 342, we may form polynomials in z with complex coefficients, also fractions whose numerators and denominators are two such polynomials without common factors. Such polynomials and fractions are functions of z in the sense of § 14. They are called *rational functions, integral* or *fractional.*

An integral function, $f(z) = a_0 z^n + \cdots + a_n$, is *continuous* at every finite point $z = \alpha$; that is, § 16, $f(\alpha)$ is finite and $\lim_{z \to \alpha} f(z) = f(\alpha)$.

A fractional function $f(z)$ is continuous except at the points where its denominator vanishes; but if β is such a point, then $\lim_{z \to \beta} | f(z) | = \infty$.

The definition of *derivative*, § 23, applies without modification to rational functions $f(z)$; and we may prove as in § 29, for positive integral values of n, that $D_z z^n = nz^{n-1}$.

348. Infinite series with complex terms. Consider the series

$$\Sigma u_n = \Sigma(a_n + ib_n) = (a_1 + ib_1) + (a_2 + ib_2) + \cdots + (a_n + ib_n) + \cdots$$

If the real series Σa_n and Σb_n converge to the sums A and B, then Σu_n converges to the sum $A + iB$. But if either Σa_n or Σb_n is divergent, so also is Σu_n.

If $\Sigma \mid u_n \mid$ is convergent, then Σu_n is convergent, and it is said to be absolutely convergent.

For $| a_n | \leqq (a_n^2 + b_n^2)^{1/2}$ and $| b_n | \leqq (a_n^2 + b_n^2)^{1/2}$. Therefore since $\Sigma(a_n^2 + b_n^2)^{1/2}$ converges, so also do Σa_n and Σb_n, §§ 168, 176, hence also Σu_n.

349. Power series in z. This name is given to series of the form $\Sigma a_n(z - c)^n$, where z is a complex variable, and c, a_0, a_1, $\cdots$ are complex constants.

350. Theorem 1. *If the power series $\Sigma a_n z^n$ converges when $z = b$, it converges absolutely when $| z | < | b |$.*

For since $\Sigma a_n b^n$ converges, all its terms are finite; that is, we can find a finite positive number, M, such that $| a_n b^n | < M$ for every n. We then have

$$| a_n z^n | = | a_n b^n | \cdot \left| \frac{z}{b} \right|^n < M \left| \frac{z}{b} \right|^n, \text{ for every } n.$$

Hence each term of $\Sigma \mid a_n z^n \mid$ is less than the corresponding term of the geometric series $\Sigma M \left| \frac{z}{b} \right|^n$ and therefore converges when $\Sigma M \cdot \left| \frac{z}{b} \right|^n$ converges, that is, when $\mid z/b \mid < 1$, or $\mid z \mid < \mid b \mid$.

351. Corollary. *If $\Sigma a_n z^n$ diverges when $z = b$, it diverges when $\mid z \mid > \mid b \mid$.*

For were $\Sigma a_n z^n$ to converge for a value $z = c$ such that $\mid c \mid > \mid b \mid$, it would also converge for $z = b$, § 350.

352. Circle of convergence. It follows from §§ 350, 351 that every $\mid z \mid$ for which a given series $\Sigma a_n z^n$ converges is less than every $\mid z \mid$ for which it diverges. Therefore, unless it never converges or always converges, there is a definite number R which is either the greatest $\mid z \mid$ for which $\Sigma a_n z^n$ converges or the least $\mid z \mid$ for which it diverges, § 320. In the z-plane, draw about O as center the circle of radius R, and call it O_R. At every point inside of O_R, $\Sigma a_n z^n$ converges absolutely, at every point outside it diverges, at a point on the circumference it may converge or diverge. We therefore call R the *radius* and O_R the *circle of convergence* of $\Sigma a_n z^n$. In case $\Sigma a_n z^n$ always converges, $R = \infty$; thus, $\Sigma z^n/n!$. In case $\Sigma a_n z^n$ converges only for $z = 0$, then $R = 0$; thus, $\Sigma z^n n!$.

The series $\Sigma a_n (z - c)^n$ converges when $\mid z - c \mid < R$, diverges when $|z - c| > R$. Hence its region of convergence is the circle C_R of radius R about the point $z = c$ as center.

Example. Draw the circles of convergence of $\Sigma (2z)^n$, $\Sigma z^n/n$, $\Sigma [z - (1 + i)]^n$.

353. Functions defined by power series in z. At each point z within its circle of convergence C_R, a given power series $\Sigma a_n (z - c)^n$ has a definite sum. Hence $\Sigma a_n (z - c)^n$ defines a function of z in C_R, § 14. This function $f(z)$ we call *the sum of $\Sigma a_n (z - c)^n$ in C_R* and write

$$f(z) = a_0 + a_1(z - c) + \cdots + a_n(z - c)^n + \cdots, \quad \mid z - c \mid < R$$

354. Theorem 2. *Let R be the radius of convergence of $\Sigma a_n z^n$, and let ρ be any positive number $< R$. The function $f(z) = \Sigma a_n z^n$ is continuous*[1] *in the circle O_ρ of radius ρ about O.*

The proof of this theorem is the same as that for real series, § 181. And by the same reasoning as for real series, § 309, we may prove the following :

355. Theorem 3. *Let the terms of the series Σu_n be functions of z which within the circle O_R are continuous and numerically less than the corresponding terms of a given positive series ΣM_n; in any circle O_ρ, $\rho < R$, the sum $f(z)$ of Σu_n is a continuous function of z.*

356. Theorem 4. *Let $f(z) = \Sigma a_n z^n$ converge within the circle O_R. The series $\Sigma n a_n z^{n-1}$, got by differentiating $\Sigma a_n z^n$ term by term, will also converge within O_R, and its sum will be $f'(z)$, the derivative of $f(z)$.*

For let z denote any fixed point within O_R, and $h = \Delta z$ any variable increment such that $|z| + |h| < R$.

Since $f(z) = \Sigma a_n z^n$ and $f(z+h) = \Sigma a_n (z+h)^n$ both converge, so also will the series got by subtracting the first from the second and dividing the result by h, namely

$$\frac{f(z+h) - f(z)}{h} = \Sigma a_n \frac{(z+h)^n - z^n}{h}$$
$$= \Sigma a_n \left[n z^{n-1} + \frac{n(n-1)}{1 \cdot 2} z^{n-2} h + \cdots \right]$$

The terms of this series are continuous functions of h and they are numerically $\leq$ the corresponding terms of the convergent positive series

$$\Sigma |a_n| \left[n |z|^{n-1} + \frac{n(n-1)}{1 \cdot 2} |z|^{n-2} |h| + \cdots \right]$$

Hence, by Theorem 3., their sum is a continuous function of h, and therefore, when $h \to 0$, it approaches as limit the value it has when $h = 0$. Hence

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \Sigma n a_n z^{n-1}$$

[1] As in § 16, we say that $f(z)$ is *continuous at* $x = c$ if $f(c)$ has a definite finite value and if $f(z) \to f(c)$ when $z \to c$; also that $f(z)$ is *continuous in a region* S, if it is continuous at all points of S. See also § 337.

Hence a function $f(z)$ defined by a power series has continuous derivatives of every order.

357. The function e^z. 1. For all finite real values of z, by § 189,

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \tag{1}$$

We make (1) our *definition* of e^z when z is a complex number. To justify so doing, it is only necessary to show that e^z as thus defined has the functional properties which characterize e^z when z is real, namely :

$$e^z \cdot e^{z_1} = e^{z+z_1} \qquad De^z = e^z \tag{2}$$

But by the rule for multiplication of series, § 300, 1., extended to two variables, it is readily shown that $[\Sigma z^n/n!][\Sigma z_1^n/n!] = \Sigma(z+z_1)^n/n!$; which gives, by (1), $e^z \cdot e^{z_1} = e^{z+z_1}$. And since $\Sigma z^n/n!$ reproduces itself when differentiated term by term, we have, by (1) and § 356, $De^z = e^z$.

2. When $z = i\theta$, θ being real, (1) becomes

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

Hence, § 190, $\qquad e^{i\theta} = \cos\theta + i\sin\theta \qquad (3)$

when $\theta = 2\pi$, (3) gives $\qquad e^{2\pi i} = 1 \qquad (4)$

Hence, by (2), $\qquad e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z \qquad (5)$

which shows that e^z is a *periodic function* with the period $2\pi i$.

3. It also follows from (3) and § 343 (1), that any complex number $a + bi$ may be expressed in terms of its absolute value $r = (a^2 + b^2)^{1/2}$ and its angle θ by the formula

$$a + bi = re^{i\theta} \tag{6}$$

4. Hence also, § 343, 3., any *vector* in the z-plane whose length is r and whose direction makes the angle θ with Ox may be written $re^{i\theta}$.

358. Logarithms. 1. Given a complex number $\alpha = r(\cos\phi + i\sin\phi)$, $r \neq 0$, it is required to find $z = x + iy$ such that $e^z = \alpha$.

The equation to be solved is

$$e^x(\cos y + i\sin y) = r(\cos\phi + i\sin\phi)$$

which means, § 341, 4.,

$$e^x \cos y = r\cos\phi, \quad e^x \sin y = r\sin\phi.$$

Hence $e^x = r$, or $x = \log r$; and $y = \phi + 2k\pi$, k being any integer.

We call z the *logarithm* of α and write it $\text{Log}\,\alpha$. Hence

$$\text{Log}\, re^{i\phi} = \log r + i(\phi + 2k\pi) \tag{1}$$

Observe that $\text{Log}\, re^{i\phi}$ is infinitely many valued. As its *principal value*, we take that for which $-\pi < \phi + 2k\pi \leqq \pi$.

2. We define z^m, when z and m are complex, by the formula

$$z^m = e^{m\,\text{Log}\, z} \tag{2}$$

Hence, except when m is rational, z^m has infinitely many values.

EXAMPLE. Since $i = e^{\pi i/2}$, we have

$$i^i = e^{i\,\text{Log}\, i} = e^{i \cdot i(\pi/2 + 2k\pi)} = e^{-(\pi/2+2k\pi)}.$$

359. The functions $\sin z$, $\cos z$. 1. By § 190, when z is real,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \tag{1}$$

We make the equations (1) our definitions of $\sin z$, $\cos z$ when z is complex. For it follows from § 357 that $\sin z$, $\cos z$ as thus defined have all the functional properties which characterize them when z is real.

Thus, § 357 (1), $e^{iz} = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$

Hence, by (1), $$e^{iz} = \cos z + i\sin z \tag{2}$$

Similarly, $$e^{-iz} = \cos z - i\sin z \tag{3}$$

Solving (2), (3), $$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{4}$$

By (4), we obtain

$$\sin^2 z + \cos^2 z = 1 \qquad \sin(z + z_1) = \sin z \cos z_1 + \cos z \sin z_1$$

and so on; also, § 356,

$$D \sin z = \cos z, \qquad D \cos z = -\sin z.$$

2. We define tan z by the formula $\tan z = \sin z/\cos z$; and cot z, sec z, cosec z similarly.

3. It follows from the second of the formulas (4) that

$$\text{arc} \sin z = \frac{1}{i} \text{Log} (iz \pm \sqrt{1 - z^2}) \tag{5}$$

For, if w satisfies the equation $\sin w = z$, we have by (4)

$$z = \frac{e^{iw} - e^{-iw}}{2\,i} = \frac{e^{2iw} - 1}{2\,ie^{iw}} \qquad \therefore\ e^{2iw} - 2\,ize^{iw} - 1 = 0$$

$$\therefore\ e^{iw} = iz \pm \sqrt{1 - z^2} \qquad \therefore\ w = \frac{1}{i} \text{Log} (iz \pm \sqrt{1 - z^2})$$

EXERCISE LXIV

1. Prove that $\cos 2\,z = \cos^2 z - \sin^2 z$ when z is complex.

2. Prove that $\arctan z = \dfrac{1}{2\,i} \text{Log} [(1 + iz)/(1 - iz)]$. Verify geometrically for the case in which z is real.

3. If $z = Re^{i\theta}$, show that $e^{iz} = e^{-R \sin \theta} \cdot e^{iR \cos \theta}$.

4. Reduce each of the following to the form $re^{i\theta}$:
(1) $e^{2+3i}e^{-1+2i}$ (2) 2^i (3) $(1 + i)^i$ (4) $(1 + \sqrt{3}\,i)^{1+i}$

5. Show that $z^{1/2}$ changes sign when z traces a closed curve which includes the origin. What happens to Log z when z traces this curve?

6. A point P, $z = re^{i\theta}$, is moving in the z-plane. Find the components of its velocity and acceleration, along and perpendicular to OP, from dz/dt and d^2z/dt^2. Compare § 272.

360. Analytic functions. 1. By substituting $x + iy$ for z, any of the functions of z thus far considered may be reduced to the form $u(x, y) + i\,v(x, y)$, where u and v denote real functions of x, y. Thus

$$z^2 = (x + iy)^2 = (x^2 - y^2) + 2\,xyi.$$

If, as in § 14, we say that w is a function of z when to each value of z there corresponds a definite value of w, then *every*

one-valued function of x, y of the form $w = u + iv$ is a function of z; for when a value of z is given, the values of x and y are given, and therefore the value of $u + iv$ is determined. Thus, if $w = x + 2\,yi$, then when $z = 3 + 5\,i$, we have $w = 3 + 10\,i$.

But nothing is gained by regarding a function $u + iv$ as a function of z unless it has a z-derivative. The functions of z considered in the preceding sections possess this property, but the function $w = x + 2\,yi$ does not. We proceed to show that it is characteristic of the functions $u + iv$ which have z-derivatives, that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

For, using z and w to denote corresponding values of $z = x + iy$ and $w = u + iv$, give z any increment $\Delta z = \Delta x + i\,\Delta y$, and let $\Delta w = \Delta u + i\,\Delta v$ denote the resulting increment of w. Suppose that for all manners of approach of Δz to 0 as limit, the ratio $\Delta w/\Delta z$ approaches one and the same limit. We then call that limit the z-derivative of w at the point z and represent it by dw/dz. If $\Delta w/\Delta z$ does not approach such a limit, we say that w has no z-derivative at the point z.

If w has a z-derivative at the point z, then the partial derivatives u_x, u_y, v_x, v_y exist at the point z and satisfy the equations (1).

For among the possible modes of approach of Δz to 0 are those for which Δz takes real values only ($\Delta z = \Delta x$) and those for which it takes pure imaginary values only ($\Delta z = i\,\Delta y$). Hence the statement that dw/dz exists at the point z implies in particular that at z the limits

$$\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} \quad \text{and} \quad \lim_{\Delta y \to 0} \frac{\Delta w}{i\,\Delta y} \quad \text{exist and are equal.}$$

By § 215, these limits exist when, and only when, u_x, v_x, u_y, v_y exist, and we then have

$$\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$$\lim_{\Delta y \to 0} \frac{\Delta w}{i\,\Delta y} = \frac{1}{i}\left[\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}\right] = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}$$

Therefore, since these limits must be equal, we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Conversely, if at the point z the partial derivatives u_x, u_y, v_x, v_y exist, are continuous, and satisfy the equations (1), *then dw/dz exists and is continuous at z. Moreover*

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{2}$$

For since u_x, u_y, v_x, v_y exist and are continuous, we have, § 216 (6),

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y$$

where ϵ_1, ϵ_2, ϵ_3, $\epsilon_4 \to 0$ when Δx, $\Delta y \to 0$.

But since u_x, u_y, v_x, v_y satisfy the equations (1), we may replace $\partial u/\partial y$ by $-\partial v/\partial x$ and $\partial v/\partial y$ by $\partial u/\partial x$. This having been done, add i times the second equation to the first, and in the result represent $\epsilon_1 + i\epsilon_3$ by ζ_1 and $\epsilon_2 + i\epsilon_4$ by ζ_2. We thus obtain

$$\begin{aligned}\Delta u + i\,\Delta v &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \zeta_1 \Delta x + \zeta_2 \Delta y \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\,\Delta y) + \zeta_1 \Delta x + \zeta_2 \Delta y\end{aligned}$$

Hence $$\frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \zeta_1\frac{\Delta x}{\Delta z} + \zeta_2\frac{\Delta y}{\Delta z}$$

When $\Delta z \to 0$, the last two terms $\to 0$; for $\left|\frac{\Delta x}{\Delta z}\right|$, $\left|\frac{\Delta y}{\Delta z}\right| \leq 1$, and $\zeta_1, \zeta_2 \to 0$.

Therefore $$\lim_{z \to 0}\frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

2. A function $f(z)$ is said to be *analytic* at the point $z = c$ if it has a continuous derivative $f'(z)$ at $z = c$. If $f(z)$ is analytic at all points of a region S, it is said to be analytic in S. If it is analytic except at certain points of S, these points are called its *singular points* in S.[1]

[1] Goursat has shown that if $f'(z)$ exists at all points of a region S, it is necessarily continuous, and therefore analytic, in S. See Goursat-Hedrick, Mathematical Analysis, Vol. II, §§ 28, 33.

Thus $1/(z^2 + 1)$ is analytic except at the singular points $z = i$ and $z = -i$. And, by §§ 354, 356, a function defined by a power series is analytic within the circle of convergence of the series.

By what has just been proved, a function $u(x, y) + iv(x, y)$ is an analytic function of z in a region S, when and only when, throughout S, the partial derivatives u_x, u_y, v_x, v_y exist, are continuous, and satisfy the equations $u_x = v_y$, $u_y = -v_x$.

In what follows, $f(z)$ will denote an analytic function, whether given in the form $f(z)$ or the form $u + iv$. In § 369 it is proved that the class of such functions is the same as that of functions $f(z)$ defined by power series.

361. The integral $\int_{AB} f(z)\,dz$. 1. Suppose that $f(z) = u + iv$ is analytic in a simply connected region R of the z-plane, § 287, 4. Let AB be any curve arc in R of the kind described in § 286. Divide AB into n parts all of which $\to 0$ when $n \to \infty$. Let δs denote any one of these parts, and δx and δy the projections of δs on Ox and Oy, and let $\delta z = \delta x + i\,\delta y$; so that $|\delta z|$ is the length of the chord of δs. Then, if z be any point of δs, we have

$$\begin{aligned}\Sigma f(z)\,\delta z &= \Sigma[u(x, y) + iv(x, y)](\delta x + i\,\delta y)\\ &= \Sigma(u\,\delta x - v\,\delta y) + i\Sigma(v\,\delta x + u\,\delta y)\end{aligned}$$

and, by § 286, when $n \to \infty$ this sum approaches the limit

$$\int_{AB}(u\,dx - v\,dy) + i\int_{AB}(v\,dx + u\,dy)$$

Hence, if we represent $\lim \Sigma f(z)\,\delta z$ by $\int_{AB} f(z)\,dz$, we have

$$\int_{AB} f(z)\,dz = \int_{AB}(u\,dx - v\,dy) + i\int_{AB}(v\,dx + u\,dy) \qquad (1)$$

2. Each of the integrals in the second member of (1) is of the type $\int P\,dx + Q\,dy$ where, as follows from § 360 (1), $\partial Q/\partial x = \partial P/\partial y$. Hence, § 288, 1., the value of $\int_{AB} f(z)\,dz$ is the same for all paths in R from A to B, and it may be represented by $\int_A^B f(z)\,dz$.

362. The integral $\int_{z_0}^{z} f(z)\,dz$. 1. It follows from § 361, 2., that the integral $\int_{z_0}^{z} f(z)\,dz$ taken on any path in R from the fixed point $z_0 = a + ib$ to the variable point $z = x + iy$ is a one valued function of this variable z. We are to prove that it is an analytic function, and that its derivative is $f(z)$.

For set
$$\int_{z_0}^{z} f(z)\,dz = P + iQ$$
where
$$P = \int_{a,\,b}^{x,\,y} u\,dx - v\,dy \qquad Q = \int_{a,\,b}^{x,\,y} v\,dx + u\,dy \qquad (1)$$

Since $\partial P/\partial x = u$, and $\partial Q/\partial y = u$, § 288, we have $\partial P/\partial x = \partial Q/\partial y$; and since $\partial P/\partial y = - v$, and $\partial Q/\partial x = v$, we have $\partial P/\partial y = - \partial Q/\partial x$.

Hence $P + iQ$ is an analytic function, § 360. Call it $F(z)$. Then, § 360 (2),
$$F'(z) = \partial P/\partial x + i\,\partial Q/\partial x = u + iv = f(z)$$

2. If $f(z) = \Sigma a_n z^n$, and $\phi(z) = \Sigma a_n z^{n+1}/(n + 1)$, then
$$\int_{z_0}^{z} f(z)\,dz = \phi(z) - \phi(z_0) \qquad (2)$$

For set $F(z) - \phi(z) = W = U + iV$. Since $D_z[F(z) - \phi(z)] = f(z) - f(z) \equiv 0$, we have U_x, V_x, U_y, V_y, all $\equiv 0$ $\therefore \Delta W = 0$, § 223 (1), $\therefore$ W is a constant, and therefore $F(z) = \phi(z) +$ const. But from this (2) follows as in § 126, 2.

363. Cauchy's theorem. *Let C denote any closed curve in the z-plane, of the kind described in § 287, and S the region within C. If the function $f(z)$ is analytic on C and in S, then*
$$\int_C f(z)\,dz = 0 \qquad (1)$$

For replacing AB by C in § 361 (1), we have
$$\int_C f(z) = \int_C (u\,dx - v\,dy) + i \int_C (v\,dx + u\,dy)$$
And since each integral in the second member is of the type $\int P\,dx + Q\,dy$ where $\partial Q/\partial x = \partial P/\partial y$, each is 0 by Green's formula, § 287, 2.

364. Corollary. *Let S be a region bounded by an outer contour C and an inner contour C_1. If $f(z)$ be analytic in S and on C, C_1, then*
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz \qquad (1)$$
where C and C_1 are supposed traced in the same sense.

For join C and C_1 by the line ad. The integral $\int f(z)\,dz$ taken along the single contour $adefdabca$ is 0 by § 363 (1). The parts along ad and da cancel each other; the part along $abca$ is $\int_C f(z)\,dz$; and that along $defd$ is $-\int_{C_1} f(z)\,dz$.

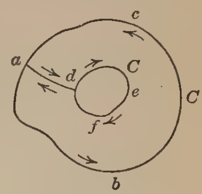

FIG. 150.

Similarly if there be two interior contours C_1 and C_2, we have

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz,$$

all three contours being traced in the same sense.

365. The integral $\int_\gamma dz/(z-c)$. 1. Let c be any point in the z-plane, and γ a circle of radius r about c as center. Then

$$\int_\gamma \frac{dz}{z-c} = 2\,\pi i \tag{1}$$

For set $z - c = re^{i\theta}$, § 357 (6). Then since r is constant,

$$dz = re^{i\theta} i\,d\theta.$$

Hence $$\int_\gamma \frac{dz}{z-c} = \int_0^{2\pi} i\,d\theta = 2\,\pi i, \quad \text{which gives (1).}$$

On the other hand, if n be any integer $\neq 1$, then, § 357 (3),

$$\int_\gamma \frac{dz}{(z-c)^n} = \frac{i}{r^{n-1}}\int_0^{2\pi} e^{(-n+1)i\theta}\,d\theta = 0 \tag{2}$$

It follows from § 364 (1) that in the integrals (1) and (2) we may replace the circle γ by any closed curve C which contains the point c.

366. Residues. It follows from § 365 (1), (2) and § 362, 2., that if c be a singular point of $f(z)$ such that, near c,

$$f(z) = \frac{a_{-m}}{(z-c)^m} + \cdots + \frac{a_{-1}}{z-c} + a_0 + a_1(z-c) + \cdots \tag{1}$$

and the contour C encloses c and no other singular point of $f(z)$, then

$$\int_C f(z)\,dz = 2\,\pi i a_{-1} \tag{2}$$

The coefficient a_{-1} is called the *residue* of $f(z)$ at $z = c$. By § 364,

If the contour C encloses one or more singular points of $f(z)$, and if ΣR_k denote the sum of the residues of $f(z)$ at these points, then

$$\int_C f(z)\, dz = 2\,\pi i \Sigma R_k \tag{3}$$

Example 1. The function $f(z) = 3/z + i/z^2 + (2 + 3\,i)/(z - 2\,i)$ has the singular points 0 and $2\,i$, with the residues 3 and $2 + 3\,i$. Hence if C encloses 0 and $2\,i$, then

$$\int_C f(z) = 2\,\pi i[3 + (2 + 3\,i)] = 2\,\pi i(5 + 3\,i)$$

367. Applications to definite integrals. The following examples illustrate the use that may be made of the theorems of §§ 363–366 in evaluating certain real definite integrals. But before turning to them, observe that

1. *If C denote a given curve arc or contour, L the length of C, and M the greatest value of $|\,f(z)\,|$ on C, then $|\int_C f(z)\, dz\,| \leqq LM$.*

For $\int_C f(z)\, dz = \lim \underset{C}{\Sigma} f(z)\, \delta z$, and $|\,\underset{C}{\Sigma} f(z)\, \delta z\,| \leqq M \underset{C}{\Sigma}\,|\,\delta z\,| \leqq ML$, since $\underset{C}{\Sigma}|\,\delta z\,| \to L$, § 361.

2. *If C be a circle of radius R about the point c as center, and M be the greatest value of $|\,f(z)(z - c)\,|$ on C, then*

$$|\int_C f(z)\, dz\,| \leqq 2\,\pi M$$

For setting $z - c = Re^{i\theta}$, we have

$$\int_C f(z)\, dz = \int_C f(z)(z - c)\frac{dz}{z - c} = i\int_0^{2\pi} f(z)(z - c)\, d\theta$$

Hence $$|\int_C f(z)\, dz\,| \leqq M\int_0^{2\pi} d\theta = 2\,\pi M$$

Therefore if $M \to 0$ when $R \to 0$ or ∞, then $\int_C f(z)\, dz \to 0$ when $R \to 0$ or ∞.

Example 1. Let $F(z) = \phi(z)/f(z)$ be a rational fraction in its lowest terms, with real coefficients, and such that the degree of $f(z)$ exceeds that of $\phi(z)$ by at least two units and all the roots of $f(z) = 0$ are imaginary. Let ΣR_k denote the sum of the residues of $F(z)$ at the roots of $f(z) = 0$ in the upper half of the z-plane. Then

$$\int_{-\infty}^{\infty} F(x)\, dx = 2\,\pi i \Sigma R_k \tag{1}$$

For draw a semicircle ABA' great enough to enclose, with $A'OA$, all the roots of $f(z) = 0$ in the upper half of the z-plane. By § 366 (3), the integral $\int F(z)\,dz$ taken along the contour $A'OABA'$ equals $2\pi i \Sigma R_k$. Hence, if $OA = R$,

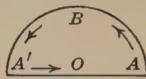

FIG. 151.

$$\int_{-R}^{R} F(x)\,dx + i\int_0^{\pi} F(Re^{i\theta})Re^{i\theta}\,d\theta = 2\pi i \Sigma R_k$$

Let $R \to \infty$. Then $\int_{-R}^{R} F(x)\,dx \to \int_{-\infty}^{\infty} F(x)\,dx$; and, $F(Re^{i\theta})R$ being a proper fraction with respect to R, $\int_0^{\pi} F(Re^{i\theta})Re^{i\theta}\,d\theta \to 0$, by § 367, 2. Hence (1).

EXAMPLE 2. Prove that $\displaystyle\int_0^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}$.

First consider the integral $\displaystyle\int \frac{e^{iz}}{z}\,dz$ taken on the contour $ABDB'A'EA$, where BDB' and $A'EA$ are semicircles with radii R and r about O as center. Since the contour encloses no singular point,

$$\int_r^R \frac{e^{ix}}{x}\,dx + \int_{BDB'} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_{A'EA} \frac{e^{iz}}{z} z = 0$$

$$\therefore 2i\int_r^R \frac{\sin x}{x}\,dx = -\int_{A'EA} \frac{e^{iz}}{z}\,dz - \int_{BDB'} \frac{e^{iz}}{z}\,dz \qquad (1)$$

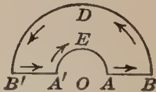

FIG. 152.

It will be shown that when $r \to 0$ and $R \to \infty$, the first term on the right $\to \pi i$, the second $\to 0$.

1. For on $A'EA$ we have $z = re^{i\theta}$, and $e^{iz} = 1 + (e^{iz} - 1)$, where $|e^{iz} - 1| = |iz + \cdots| \to 0$ when $r \to 0$.

$$\text{Hence} \int_{A'EA} \frac{e^{iz}}{z}\,dz = \int_{\pi}^{0} i\,d\theta + \int_{\pi}^{0} (e^{iz} - 1)i\,d\theta \to -\pi i, \text{ when } r \to 0.$$

2. On BDB' we have $\quad z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$

$$\therefore e^{iz} = e^{-R\sin\theta} \cdot e^{iR\cos\theta}$$

Hence

$$\left|\int_{BDB'} \frac{e^{iz}}{z}\,dz\right| = \left|\int_0^{\pi} e^{-R\sin\theta} \cdot e^{iR\cos\theta}\, i\,d\theta\right| < \int_0^{\pi} e^{-R\sin\theta}\,d\theta = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta.$$

But when $\theta < \frac{\pi}{2}$ we have $\dfrac{\sin\theta}{\theta} \geqq \dfrac{2}{\pi} \quad \therefore R\sin\theta \geqq \dfrac{2R\theta}{\pi}$

$$\text{Hence} \quad \int_0^{\pi/2} e^{-R\sin\theta}\,d\theta < \int_0^{\pi/2} e^{-\frac{2R\theta}{\pi}}\,d\theta = \frac{\pi}{2R}(1 - e^{-R}) \quad \text{which} \to 0$$

when $R \to \infty$,

Therefore $\int_{BDB'} \frac{e^{iz}}{z} dz \to 0$ when $R \to \infty$. Therefore, by (1),

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

368. Cauchy's formula. *Let $f(z)$ denote a function which is analytic at all points on and within a given closed curve C. The value of $f(z)$ at any point $z = c$ within C may be expressed in terms of its values on C by the formula*

$$f(c) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - c} \tag{1}$$

For let γ denote a circle of radius r about c and within C. Then, since $f(z) = f(c) + [f(z) - f(c)]$, we have, by § 365, (1)

$$\int_C \frac{f(z)\,dz}{z-c} = \int_\gamma \frac{f(z)\,dz}{z-c} = 2\pi i f(c) + \int_\gamma \frac{f(z) - f(c)}{z - c} dz$$

But $f(z)$ being continuous, if any ϵ be assigned we can take r small enough to make $|f(z) - f(c)| < \epsilon$ at all points on γ and therefore, § 367, 2., $\left|\int_\gamma \frac{f(z) - f(c)}{z - c} dz\right| < 2\pi\epsilon$.

Therefore, since $\int_C \frac{f(z)\,dz}{z - c}$ and $2\pi i f(c)$ are constants, and it has been proved that they differ by less than any number $2\pi\epsilon$ that may be assigned, they are equal.

369. Taylor's series. *Let $f(z)$ be analytic at the point $z = c$ and in its neighborhood; let R be the distance from c to the nearest singular point or points of $f(z)$; and let C_R be the circle of radius R about c as center. There exists a power series $\Sigma a_n(z - c)^n$, and but one, such that $f(z) = \Sigma a_n(z - c)^n$ at all points z within C_R.*

For let C be the circle with center c and any radius $AB < R$. If t be any point within C, then, by § 368, (1),

$$f(t) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - t} \tag{1}$$

Since $\qquad z - t = (z - c)\left[1 - \dfrac{t - c}{z - c}\right] \qquad$ we have

$$\frac{1}{z-t} = \frac{1}{z-c} + \frac{t-c}{(z-c)^2} + \cdots + \frac{(t-c)^{n-1}}{(z-c)^n} + \frac{(t-c)^n}{(z-t)(z-c)^n} \tag{2}$$

Substituting this expression for $1/(z-t)$ in (1), we obtain

$$f(t) = a_0 + a_1(t-c) + \cdots + a_{n-1}(t-c)^{n-1} + R_n \tag{3}$$

where

$$a_0 = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-c}, \quad a_1 = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-c)^2}, \quad \cdots,$$

$$R_n = \frac{1}{2\pi i}\int_C \frac{f(z)(t-c)^n\,dz}{(z-t)(z-c)^n}$$

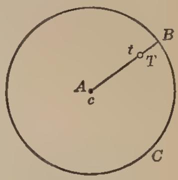

Fig. 153.

Here $a_0 = f(c)$; a_1, a_2, $\cdots$ have definite finite values; and if M denote the greatest value of $|f(z)|$ on C, then, § 367, 2.,

$$|R_n| < \frac{M \cdot AB}{TB}\left(\frac{AT}{AB}\right)^n, \text{ and therefore} \rightarrow 0 \text{ when } n \rightarrow \infty$$

We therefore have, since t is any value of z within C_R,

$$f(z) = a_0 + a_1(z-c) + \cdots + a_n(z-c)_n + \cdots, \ |z-c| < R \tag{4}$$

In § 186 it was proved, for functions of real variables, that if a power series exists of which such a function is the sum, the series is a Taylor series. But the proof depended solely on the theorem on the differentiation of a power series term by term, — which, by § 356, holds good for series in the complex variable also. Hence (4) is a Taylor series : that is, for $|z-c| < R$,

$$f(z) = f(c) + f'(c)\frac{(z-c)}{1} + \cdots + f^{(n)}(c)\frac{(z-c)^n}{n!} + \cdots \tag{5}$$

370. General conclusions. 1. It follows from the theorems of § 369 and § 356 that if $f(z) = u + iv$ is analytic in a region S, then u and v have continuous partial derivatives of every order within S.

2. The §§ 349–356 also show that, apart from its geometrical interpretations, the entire theory of operations with functions defined by power series which was developed for

real series in §§ 191, 299–310, holds good for complex series also. In particular it follows from § 301 that

(1) *If w is an analytic function of z at the "point" (z_0, w_0) and in its neighborhood, and if $dw/dz \neq 0$ at (z_0, w_0), then z is an analytic function of w at (z_0, w_0) and in its neighborhood.*

(2) *An algebraic equation $F(z, w) = 0$ defines w as an analytic function of z at and in the neighborhood of any point (z_0, w_0) where F is 0 and $\partial F/\partial w \neq 0$.*

EXERCISE LXV

1. Show that the radii of the circles of convergence of the series for $1/(z^2 + 1)$ and $1/(z^4 + 1)$ in powers of $z + 1$ are $\sqrt{2}$ and $(2 - \sqrt{2})^{1/2}$.

2. Prove that the Maclaurin series for $\sec x$ converges when $|x| < \pi/2$.

3. Show that the series for $(x + 2)/(x^2 - 2x + 2)$ in powers of $x + 1$ converges when x is between $-\sqrt{5} - 1$ and $\sqrt{5} - 1$.

4. Show that if $u + iv$ is analytic, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

5. When $u + iv$ is analytic, u and v are called *conjugate functions.* Show that if u is a given function such that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, then the integral $v = \displaystyle\int_{a,\,b}^{x,\,y} - \frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ is a function conjugate to u. Find a function conjugate to $u = 2xy + y$.

6. Let $z = re^{i\phi}$. Show that $\int_1^z (1/z)\,dz = \log r + i\phi = \log z$, if the path of integration be from $A(1, 0)$ to $B(r, 0)$ on Ox, and then from B to z on the circle of radius r about O. Hence show that $\dfrac{d}{dz}\log z = \dfrac{1}{z}$.

7. Prove that $\log(1 + z) = z - z^2/2 + z^3/3 - \cdots, |z| < 1$.

8. Let $F(z) = \phi(z)/f(z)$ be a proper fraction in its lowest terms, and let c be a simple root of $f(z) = 0$. Show that the residue of $F(z)$ at $z = c$ is $\phi(c)/f'(c)$.

9. Find $\int_C [z^2/(z^3 + 1)]\,dz$, when C is the circle of radius 2 about O.

10. By the method of § 367, Ex. 1, show that $\int_{-\infty}^{\infty} 1/(x^4 + 1)dx = \pi/4$.

11. By the same method find

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 4}, \quad \int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)}, \quad \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^2}.$$

12. Show that if $f(z) = a_0(z - z_1)(z - z_2) \cdots (z - z_n)$, then

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_n}.$$

Hence show that the number N of the roots $z_1, z_2, \cdots$ of $f(z) = 0$ within a contour C (a multiple root of order r being counted r times) is given by

$$N = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz.$$

13. Supply the details of the following proof that every algebraic equation $f(z) = 0$ has a root.

(1) Let C be a circle of radius R about O. By § 367, 2., if P and Q be polynomials in z of degrees $n - 2$ and n, then $\lim\limits_{R\to\infty} \int_C (P/Q)\, dz = 0$.

(2) Hence, if $f(z)$ be of degree n, then $\lim\limits_{R\to\infty} \int_C \left[\frac{f'(z)}{f(z)} - \frac{n}{z}\right] dz = 0$.

(3) But $\int_C (n/z)\, dz = 2n\pi i$, § 365. Hence, § 363, $f'(z)/f(z)$ has at least one singular point in the z-plane, and therefore $f(z)$ at least one zero point.

14. Using the contour in Fig. 151, show that if $b > 0$,

$$\int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{x - ib} = 2\pi i e^{-b}, \quad \int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{x + ib} = 0, \quad \therefore \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + b^2} dx = \frac{\pi e^{-b}}{b}.$$

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